# Total Cross Section of $e^+e^- \to \mu^+\mu^-$ Annihilation
# Semester 1 MPhys Project Report


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**Abstract**

In collision events, fermions may spontaneously radiate a boson, which can either be reabsorbed or emitted. These processes are radiative corrections which, in superposition with their leading order counterpart, results in corrections to cross-sections proportional to $\log(\frac{s}{a^2})\log(\frac{s}{b^2})$, where $s$ is the centre of mass energy squared, and $a, b$ are often related to the masses of the particles involved or an infrared (IR) regulator. This study derives these corrections in quantum electrodynamics and the electroweak theory. We find that the corrections are generally of the aforementioned form. We find that in the quantum electrodynamic case, these are IR divergent, but cancel under the scrutiny of an inclusive cross-section. In the electroweak case, we find only a partial cancellation, however the residual is IR finite, so does not pose an offence to experimental calculations.

# 1 Introduction

## 1.1 Radiative Corrections at the TeV Scale

Today, much of modern physics is concerned with the high energy frontier, where large colliders probe the TeV scale. At this energy scale, much larger than the rest mass of any standard model boson, higher order corrections to the cross-sections for scattering and annihilation processes contribute greatly. Scattering amplitudes inclusive radiative effects, the soft emission of bosons (Bremsstrahlung) or loops with soft bosons at vertices, often contain infrared (IR) divergences arising from singular denominators in the fermion propagator(s). These IR divergences lead to unphysical cross-sections containing infinities as the momentum of the soft (massless) boson goes to zero [1]. However, as discovered by Bloch and Nordsieck, consideration of an *inclusive cross-section*, one that includes the summation of both soft Bremsstrahlung *and* vertex corrections, provides a cancellation of their IR behaviour to all orders in quantum electrodynamics (QED) [2]. After cancellation of these divergences, a single scalar correction to the cross-section, proportional to $\log^2(s)$ (so called *Sudakov double logs*), is found, where $s$ is the centre-of-mass energy squared [3].

Aside, these effects arise not only in QED but also individually in quantum chromodynamics (QCD) and electroweak physics. In QCD similar IR divergences to QED are found, however their cancellation is more complex. The recovery of finite cross-sections is fulfilled after summing over the real emission, vertex loops, *and* all colour states of the particle beam [4, 5]. This additional summation of colour states is permissible in an experimental context because the colour confinement of the strong force mandates that it is impossible to determine the colour content of a hadron. Since it is impossible to prepare a monochromatic particle beam, we assume the colour composition of the beam to be randomly distributed, and thus in the large particle limit, it is acceptable to average over all possible initial colour states.

In the electroweak case, summing over only the real emission and loop corrections is not sufficient to produce a finite correction, like QCD, however one cannot simply sum over all possible initial lepton states similarly to QCD. Although the Sudakov logs in the electroweak calculations do cancel over a homogeneous beam of leptons, as it is possible to prepare a beam pure in a given lepton species, the Sudakov logs do not always cancel and only under assuming there is one cut-off parameter, the $W^{\pm}$ or $Z^0$ mass, can finite cross-sections be recovered[6]. At the TeV scale, Sudakov effects contribute around 10% to the cross-section of relevant processes.

## 1.2 This Report

This report details an investigation into the origin of the Sudakov logs, both in pure QED and the electroweak sector. Our approach is phenomenological in nature; we derive the Sudakov logs by explicitly computing the forms of some next leading order corrections to the cross-sections for selected scattering and annihilation processes. Namely, we compute the corrections to scattering processes in QED and the electroweak theory arising from soft Bremsstrahlung and vertex loops. Using crossing-symmetry (Section 2.3), we also prove this is the same for annihilation.

In the QED case, we show the explicit cancellation of the IR divergences and ponder the physical implications of this result. Thereafter, we provide a physical prediction based on the $1^{\text{st}}$ order correction to the cross-section.

In the electroweak case, we show that cancellation of the corrections is not complete, but that due to the nature of the massive bosons mediating the weak force, there are no IR divergences, and so the correction to the cross-section is finite. We also point out that due to the nature of the electroweak theory, it could be possible that the lack cancellation of is specifically due to the lack of a counterpart emission process for a $W^{\pm}$-boson in a vertex loop.

## 1.3 Conventions

The convention in this report is that we take the spacelike components of the Minkowski metric to be negative,

$$\eta_{\mu\nu} = \text{Diag}(1, -1, -1, -1). \tag{1}$$

We will also use natural units,

$$\hbar = c = k_B = 1. \tag{2}$$

For Feynman diagrams, time runs horizontally from left to right, and the Feynman gauge, $\xi = 0$, is used implicitly to simplify propagator denominators.

Vectors without a bold typeface are 4-vectors, and those with **bold** are 3-vectors.

# 2 Theory

In this section, we explore the necessary theory which underpins the derivations of Sudakov logs in the following sections. Attention is paid to the formulation of scattering amplitudes which forms the basis of our phenomenological approach. We also investigate the construction of cross-sections from scattering amplitudes by integrating over the relevant phase-space and introducing *Mandelstam variables* into the expressions to encode the kinematics of the collisions. We will also discuss *crossing symmetry*, which allows us to switch between scattering and annihilation amplitudes. We give attention to the introduction of *Feynman parameters*, which streamlines working with multiple propagator denominators. Penultimately, we will briefly detail some important integrals from the study of classical radiation from an electron due to a potential, which simplifies many of the quantum results later in the document. Finally, we give a brief overview of PaVe (Passarino-Veltman) functions, which act as an alternative to Feynman parameters for evaluating loop integrals, amongst other applications.

## 2.1 Scattering Amplitudes

The construction of scattering amplitudes is well-documented, see [1, 7], so we only briefly detail the necessary material here.

The Feynman rules for QED and EW theory are given in Appendix A. Construction of scattering amplitudes from the Feynman rules follows a simple scheme. Firstly, one constructs a Feynman diagram of the process under investigation, for example Figure 1 - the simplest scattering process in QED.

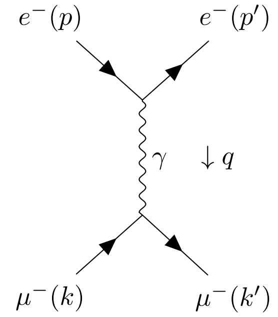


Figure 1: The tree-level QED Feynman diagram for the scattering of an electron off a muon, or vice versa. The incoming momentum for the electron (muon) is $p$ ($k$), and likewise the outgoing momentum is $p'$ ($k'$). The momentum transfer, $q$, is from the electron to the muon.

It is then possible to construct the scattering amplitude by inserting the relevant expressions for the elements within the Feynman diagram in the correct order. Beginning at an outgoing fermion (incoming anti-fermion), an anti-spinor (spinor) is collected. Then following backwards along the line, one picks up the necessary propagators and vertex couplings, finally ending with a spinor (anti-spinor). The remaining boson propagators are then inserted into the expression. For Figure 1, the scattering amplitude is given by

$$\begin{aligned} i\mathcal{M} &= [\bar{u}(p')(ie\gamma^{\mu})u(p)][\bar{u}(k')(ie\gamma^{\nu})u(k)] \\ &\times \left(\frac{-ig^{\mu\nu}}{q^2 + i\varepsilon}\right), \end{aligned} \tag{3}$$

where it can be seen that the first term in brackets corresponds to the electron line, the second corresponds to the muon line, and the final term is the photon propagator connecting the two.

Much of the work associated with scattering or annihilation processes is the extensive Dirac algebra involved in simplifying the expression for $i\mathcal{M}$ or $|\mathcal{M}|^2$. Appendix B gives some useful relations for dealing with the associated Dirac algebra.

## 2.2 Cross-Sections

We now look at the construction of cross-sections from the amplitudes, $i\mathcal{M}$. The master formula for producing cross-sections from amplitudes is

$$\begin{aligned} \sigma = \mathcal{F}\prod_f \left(\int \frac{d^3\boldsymbol{p}_f}{(2\pi)^3(2E_f)}\right)\overline{|\mathcal{M}|}^2(2\pi)^4 \\ \times\, \delta^{(4)}\Big(\sum_f p_f - p_1 - p_2\Big), \end{aligned} \tag{4}$$

where $\mathcal{F}$ is the flux scaling, given by $\mathcal{F} = 1/(4E_aE_b|\boldsymbol{v}_a - \boldsymbol{v}_b|)$ and is approximately equal to $1/2s$ when $s \gg m_am_b$ [8]. The 4D delta-function enforces the on-shell condition for real particles.

The bar over the $\overline{|\mathcal{M}|}^2$ means that $|\mathcal{M}|^2$ has been suitably averaged over the spin degrees-of-freedom of the particles, and/or any other remaining degrees-of-freedom.

## 2.3 Crossing Symmetry

We may often wish to switch from discussing a scattering event to an annihilation event. Our motivation for this not only follows from a general interest in discussing both types of event but also because scattering events are generally easier to work with. Often with annihilation events, there are additional imaginary components which can complicate the algebra.

By inspecting Figure 1, we see that rotating the diagram 90° anti-clockwise converts the diagram into an annihilation process, with an incoming electron and position, and outgoing muon and antimuon. This symmetry extends more generally than just to the Feynman diagram, and is a symmetry of the S-matrix itself, representative of a change of variables which switches any amplitude from describing a scattering event to an annihilation one, or vice versa. This variable change is given by

$$p' \to -p', \qquad k \to -k, \tag{5}$$

and the other variables stay the same.

This change of variables can be easily remembered by considering which particles go to antiparticles (or antiparticles to particles) when the relevant Feynman diagram is rotated. It is these particles/antiparticles whose momentum contains a sign change.

Although it is possible, in principle, to make this change at any point in the derivation, we should aim to restrict this change to either the start or end of a derivation to minimise the chance of mistakes.

## 2.4 Mandelstam Variables

*Mandelstam variables* provide an easy way to encode the kinematics of a collision event into a set of variables which can entirely replace the 4-momenta in the cross-section, square-amplitude, or other relevant expressions [1]. One key characteristic of Mandelstam variables is that they are entirely Lorentz invariant, so reduce the mathematical complexity associated with considering different frames of reference.

For the scattering of two particles with incoming 4-momenta $p, k$ and outgoing momenta $p', k'$, Mandelstam gives three variables $s, t, u$ such that

$$\begin{aligned} s &= (p+k)^2 = (p'+k')^2, \\ t &= (p-p')^2 = (k-k')^2, \\ u &= (p-k')^2 = (k-p')^2, \end{aligned} \tag{6}$$

where $s$ is the square of the invariant mass, and $t$ is the square of the momentum transfer. The symbols $s, t, u$ also refer to the *channels* of different Feynman diagrams/scattering events. $s$-channel refers to an annihilation event (a timelike channel), $t$-channel refers to a simple scattering event (a spacelike channel), and $u$-channel refers to a scattering event with the roles of $p', k'$ interchanged.

The inner products of the four momenta can also be parametrised in terms of the Mandelstam invariants,

$$\begin{aligned} p^\mu k_\mu &= \frac{s - m_p^2 - m_k^2}{2}, \\ p^\mu p'_\mu &= \frac{m_p^2 + m_{p'}^2 - t}{2}, \\ p^\mu k'_\mu &= \frac{m_p^2 + m_{k'}^2 - u}{2}. \end{aligned} \tag{7}$$

The invariants can also reconstruct the total mass of the particles in their own rest frames,

$$s + t + u = m_p + m_k + m_{p'} + m_{k'}. \tag{8}$$

In the relativistic limit, where $E^2 \approx \boldsymbol{p}^2$, the expressions for $s, t, u$ are relaxed,

$$\begin{aligned} s &= 2p^\mu k_\mu = 2p'^\mu k'_\mu, \\ t &= -2p^\mu p'_\mu = -2k^\mu k'_\mu, \\ u &= -p^\mu k'_\mu = -2k^\mu p'_\mu, \end{aligned} \tag{9}$$

which should be clear from taking (7) with $E \gg m_i$, so $m_i \to 0$.

## 2.5 Feynman Parameters

With each additional propagator in the amplitude, there is an additional factor in the denominator which complicates the handling of the denominator. Feynman introduced a method to reform the denominator into a single polynomial which is much easier to handle. Although not unique to simplifying loop integrals, the use of Feynman parameters is often associated with the computation of amplitudes for processes involving loops, as it provides an easy way of integrating over the loop momenta. After performing a change of variables to the Feynman parameters once they have been determined, the loop integral is spherically symmetric and therefore easy to compute.

There are numerous identities for different denominators of different forms. However, the identity which is of most use to this report is

$$\begin{aligned} \frac{1}{A_1 A_2 \ldots A_n} &= \int_0^1 dx_1 \ldots dx_n \delta\left(\sum x_i - 1\right) \\ &\quad \times \frac{(n-1)!}{[x_1 A_1 + x_2 A_2 + \ldots + x_n A_n]^n}, \end{aligned} \tag{10}$$

where $A_i$ are the propagator denominators, and $x_i$ are the corresponding Feynman parameters. The Dirac delta function in the integral gives a useful relation to use in the change of variables, explicitly $\sum x_i = 1$.

The Feynman parameters reduce the denominator to the form

$$D = l^2 - \Delta, \tag{11}$$

after a Wick rotation, where $l$ is a 4-momentum to be determined. In many cases, where the Feynman parameters are used to simplify loop integrals, the integrand is a linear combination of fractions with numerators composed of either 1 or $l^2$ and denominators $D^m$. [1] gives two standard integrals for use in this situation. They are

$$\begin{aligned} &\int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - \Delta)^m} \\ &= \frac{i(-1)^m}{(4\pi)^2} \frac{1}{(m-1)(m-2)} \frac{1}{\Delta^{m-2}}, \end{aligned} \tag{12}$$

and

$$\begin{aligned} &\int \frac{d^4 l}{(2\pi)^4} \frac{l^2}{(l^2 - \Delta)^m} \\ &= \frac{i(-1)^{m-1}}{(4\pi)^2} \frac{2}{(m-1)(m-2)(m-3)} \frac{1}{\Delta^{m-3}}. \end{aligned} \tag{13}$$

## 2.6 Classical Radiation

In anticipation of some of the results we derive from quantum field theory, we wish to briefly study the form of the energy radiated from a classical electron due to a potential kick. In summary of the system setup, we consider an electron which is kicked at $x = 0$, changing the momentum from $p$ to $p'$. As derived in [1], the total energy radiated in this system is given by

$$E_{\text{rad}} = \int \frac{d^3 k}{(2\pi)^3} \sum_{\lambda=1,2} \frac{e^2}{2} \left| \boldsymbol{\epsilon}_\lambda(\boldsymbol{k}) \cdot \left( \frac{\boldsymbol{p'}}{k \cdot p'} - \frac{\boldsymbol{p}}{k \cdot p} \right) \right|^2. \tag{14}$$

Using the completeness relation for polarisation vectors (Appendix B), an alternate expression is derived,

$$E_{\text{rad}} = \int \frac{d^3k}{(2\pi)^3} \frac{e^2}{2} \left( \frac{2p \cdot p'}{(k \cdot p')(k \cdot p)} - \frac{m^2}{(k \cdot p')} - \frac{m^2}{(k \cdot p)} \right). \tag{15}$$

Choosing the frame in which $p^0 = p'^0 = E$,

$$k^\mu = (k, \boldsymbol{k}), \quad p^\mu = E(q, \boldsymbol{v}), \quad p'^\mu = E(1, \boldsymbol{v'}), \tag{16}$$

allows the expression of $E_{\text{rad}}$ in terms of the differential intensity $\mathcal{I}(\boldsymbol{v}, \boldsymbol{v'})$,

$$E_{\text{rad}} = \frac{e^2}{(2\pi)^2} \int dk \mathcal{I}(\boldsymbol{v}, \boldsymbol{v'}), \tag{17}$$

where $\mathcal{I}(\boldsymbol{v}, \boldsymbol{v'})$ is given by

$$\mathcal{I}(\boldsymbol{v}, \boldsymbol{v'}) = \int \frac{d\Omega_{\hat{k}}}{4\pi} \left( \frac{2(1 - \boldsymbol{v} \cdot \boldsymbol{v'})}{(1 - \hat{k} \cdot \boldsymbol{v})(1 - \hat{k} \cdot \boldsymbol{v'})} - \frac{m^2}{E^2(1 - \hat{k} \cdot \boldsymbol{v'})^2} - \frac{m^2}{E^2(1 - \hat{k} \cdot \boldsymbol{v})^2} \right). \tag{18}$$

We can see trivially that this integral is divergent in two places, $\hat{k} \cdot \boldsymbol{v}, \hat{k} \cdot \boldsymbol{v'} \to 1$, which corresponds to radiation closely parallel to the incoming and outgoing electron direction. Since the major components of this integral are split into these two directions, we may approximate the integral as the sum of two components in these directions. This approximation can be written explicitly as

$$\mathcal{I}(\boldsymbol{v}, \boldsymbol{v'}) \approx \int_{\hat{k} \cdot \boldsymbol{v} = \boldsymbol{v'} \cdot \boldsymbol{v}}^{\cos\theta = 1} \frac{(1 - \boldsymbol{v} \cdot \boldsymbol{v'}) d\cos\theta}{(1 - v\cos\theta)(1 - \boldsymbol{v} \cdot \boldsymbol{v'})} + \int_{\hat{k} \cdot \boldsymbol{v'} = \boldsymbol{v'} \cdot \boldsymbol{v}}^{\cos\theta = 1} \frac{(1 - \boldsymbol{v} \cdot \boldsymbol{v'}) d\cos\theta}{(1 - v'\cos\theta)(1 - \boldsymbol{v} \cdot \boldsymbol{v'})}. \tag{19}$$

Performing these integrals, and letting the momentum transfer be $q$, we find

$$\mathcal{I}(\boldsymbol{v}, \boldsymbol{v'}) \approx 2 \log \left( \frac{-q^2}{m^2} \right), \tag{20}$$

which will be a very useful result.

### 2.7 PaVe 3-Point Integrals

In 1979, Passarino and Veltman introduced a method for evaluating loop integrals by means of substitution, which becomes very relevant when dealing with complex Feynman diagrams, as will be the case in the final section of this report. In their papers, [9, 10], they detail numerous methods for evaluating integrals of varying nature, however, we only require the so called $C$ integrals, which are used to evaluate loop integrals with three propagators.

The $n$ dimensional $C$ integrals are given by

$$C_0; C_\mu; C_{\mu\nu}; C_{\mu\nu\alpha}(p, p', m_1, m_2, m_3) = \frac{1}{i\pi^2} \int d^n k \frac{1; k_\mu; k_\mu k_\nu; k_\mu k_\nu k_\alpha}{D} \tag{21}$$

where

$$D = (k^2 + m_1^2)((k+p)^2 + m_2^2)((k+p+p')^2 + m_3^2), \tag{22}$$

and the semi-colon notation means that $C_0$ is the case when 1 is the numerator, $C_\mu$ when $k_\mu$ is the numerator, etc.

Since the numerator of the integrand of a vertex loop integral can simplified and expanded such that it is a linear combination of $1, k_\mu, \ldots$, it is possible to substitute the $1, k_\mu, \ldots$ terms for the known results of (21), which we give below[1]:

$$C_\mu = pC_{11} + p'C_{12}, \tag{23}$$

$$C_{\mu\nu} = p_\mu p_\nu C_{21} + p'_\mu p'_\nu C_{22} + \{pp'\}_{\mu\nu} C_{23} + \delta_{\mu\nu} C_{24}. \tag{24}$$

The curly bracket notation $\{pp'\}_{\mu\nu}$ corresponds to a symmetric tensor product, a bilinear combination of the momenta,

$$\{pk\}_{\mu\nu} = \frac{1}{2}(p_\mu p'_\nu + p_\nu p'_\mu). \tag{25}$$

The $C_{ij}$ terms have been explicitly evaluated in [11], which allows for further substitution. Re-explaining this slightly, this means that we ought to be able to evaluate a loop integral by expanding the numerator into linear combinations of $1, k_\mu, \ldots$ and substituting the corresponding expressions for $C_0, C_\mu, \ldots$.

## 3 Radiative Corrections in QED

Now we begin our study of the derivation of the Sudakov logs introduced earlier. In this section, we explore the effects of radiative corrections to scattering in QED. We will first discuss changes to the amplitude and cross-section of a leading order process when a soft photon is radiated from the electron in a $e^-\mu^- \to e^-\mu^-$ event. Following this discussion, we will explore similar changes made by the radiation of a photon followed by its absorption after a hard process, forming a loop around the vertex. We will critically compare the similarities between the results of each case, and speculate about any physical meaning we can derive. Finally, we conclude the section by providing an experimental prediction using this theory.

[1]The $C_{\mu\nu\alpha}$ case is not relevant to this study, but can be found in Appendix E of [9]

## 3.1 Soft Bremsstrahlung

We wish to find an expression for the cross-section of a hard process with the real emission of a soft photon before or after the hard event. Consider Figure 2 which details such a process. In this event, two fermions $f$ and $F$ scatter off each other and a single photon is radiated from the incoming $f$. A full quantum mechanical description of this process would require that we also consider the superposition of this diagram with the diagram where the photon is radiated after the hard event.

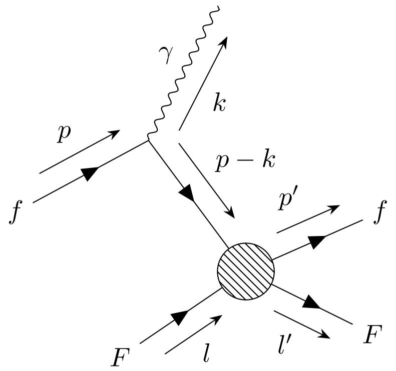


Figure 2: Scattering of two fermions $f, F$. Fermion $f$ has incoming momentum $p$, and outgoing momentum $p'$. Fermion $F$ has incoming momentum $l$ and outgoing momentum $l'$. The (hard) scattering process is represented by the *blob*. A photon with outgoing momentum $k$ is radiated from $f$ before the hard process.

Since the photon vertex does not explicitly affect $F$ or anything within the blob with the radiated photon, we can group the entire $F$ fermion line and the blob into one amplitude, $\mathcal{M}_0$, the process without any radiative emission. With this in mind, we construct the amplitude for Figure 2 and its counterpart using the normal Feynman rules. The amplitude is given by

$$\begin{aligned} i\mathcal{M} = \overline{u}(p')\Big(&\mathcal{M}_0(p', p-k)\frac{i(\not{p} - \not{k} + m)}{(p'-k)^2 - m^2} \\ &\times (-ie\gamma^\mu)\epsilon^*_\mu(k) + (-ie\gamma^\mu)\epsilon^*_\mu(k) \\ &\times \frac{i(\not{p}' + \not{k}' + m)}{(p'+k)^2 - m^2}\mathcal{M}_0(p'+k, p)\Big)u(p). \end{aligned} \tag{26}$$

This expression is then simplified significantly before considering what next major step to take. The first thing to note is that since we wish to investigate IR divergences, we can take the soft limit $k$ as this is the region which will produce the IR divergence. Since $k$ is now small with respect to the momentum transfer ($|\boldsymbol{k}| \ll |\boldsymbol{p} - \boldsymbol{p'}|$), we can make some changes to the amplitude of the hard process. We find that $\mathcal{M}_0(p', p-k) \approx \mathcal{M}_0(p'+k, p) \approx \mathcal{M}_0(p, p')$, and so we can anticipate factorising out $\mathcal{M}_0(p, p')$. Additionally, this means that the $\not{k}$ term is negligible in the propagators,$\not{p} + \not{k} + m \approx \not{p} + m$.

Using the anticommutation relations for Dirac matrices in addition to the Dirac equation (see Appendix B), we see that some terms in the numerators of (26) will simplify,

$$(\not{p} + m)\gamma^\mu \epsilon^*_\mu(k)u(p) = 2p^\mu \epsilon^*_\mu u(p), \tag{27}$$

and similarly,

$$\overline{u}(p')\gamma^\mu\epsilon^*_\mu(\not{p}' + m) = \overline{u}(p')2p'^\mu\epsilon^*_\mu. \tag{28}$$

Using the Mandelstam variables from Section 2.4, we simplify the denominators of the propagators by converting them from differences of momentum squared, to inner products of momenta. The simplifications follow from (6) and (9) which give

$$(p-k)^2 - m^2 = -2p\cdot k, \quad (p'+k)^2 - m^2 = 2p'\cdot k. \tag{29}$$

The use of the above simplifications to (26), in addition to the factorising of $\mathcal{M}_0$ as hinted, give the following amplitude

$$\overline{u}(p')\mathcal{M}_0(p', p)u(p)e\left[\frac{p'\cdot\epsilon^*(k)}{p'\cdot k} - \frac{p\cdot\epsilon^*(k)}{p\cdot k}\right]. \tag{30}$$

At this stage, we start to see the implications of soft radiation. Inspecting (30), there are two major components: an anti-spinor-amplitude-spinor term and a multiplicative factor. Looking closer at the spinor and amplitude term, we see that this is actually the form of the amplitude for Figure 2 *without* the soft Bremsstrahlung correction. This implies that the rest of (30), which does not rely on the spinor and amplitude term, must contain complete information about the radiative correction. What is more, on inspection of the remaining term, explicitly

$$e\left[\frac{p'\cdot\epsilon^*(k)}{p'\cdot k} - \frac{p\cdot\epsilon^*(k)}{p\cdot k}\right], \tag{31}$$

it is clear to see that this term is scalar, since all vectors are dotted with another vector, and all remaining terms, such as $e$, are scalar themselves. This result is exceptionally powerful since we can infer that generally any result known for $\mathcal{M}_0$ can be corrected by (31), to include the effect of soft Bremsstrahlung.

Now, the rest of this subsection is dedicated to the investigation of the consequences of this fact, namely the calculation of the cross-section.

To calculate the cross-section, we need only consider which aspects of the master cross-section equation, (4), will affect (31). Looking at (4), we see that there are multiple terms which will affect the scale factor derived. Namely, (4) demands that the matrix element not only be squared but that it also averages over all remaining degrees-of-freedom. In this case, the remaining degrees-of-freedom are the two polarisation states of the photon. Additionally, we must introduce our own caveat aside from the cross-section integral, that (31) is only a point in the phase-space since the momentum $k$ is yet to be determined, so we integrate in a Lorentz invariant manner over $k$ to account for all allowed $k$.

Putting this together, the correction to the *cross-section* is therefore

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{2k}\sum_{\lambda=1,2} e^2\left|\frac{p'\cdot\epsilon^{(\lambda)}}{p'\cdot k}-\frac{p\cdot\epsilon^{(\lambda)}}{p\cdot k}\right|^2, \tag{32}$$

where it is possible to see that each of the aforementioned actions have been implemented - (31) has been squared, is averaged over the polarisation states, $\lambda$, and is integrated over the phase-space, where we have used textbook normalisation[2].

The integrand of (32) can also be thought of as the differential probability, $d\mathcal{P}$, of radiating a single photon. However, this has problematic implications. Looking at the denominator of (32), it can be noticed that this is a divergent integral as $k \to 0$. This is especially problematic since this is the region we have defined our problem to be valid within. This would suggest that as the photon momentum tends to zero, the probability of radiating a single photon tends to infinity. This is clearly wrong, but we can restore some dignity to $d\mathcal{P}$ by considering it the number of photons radiated. In which case, as $k \to 0$, the number of soft photons radiated tends to infinity.

This is not an excellent prediction, so we now pay careful attention to the limits of the integral to determine a way to let the integral behave well. We can determine an upper limit simply by enforcing that $k$ is soft, and therefore we use the momentum transfer, $|q|$ as the upper limit. For the lower limit, we may give the photon a small fictitious mass, $\mu$, which serves to cut off the integral. This introduction of a fictitious mass is done in the hope that we find a way in the future to remove any and all dependence on $\mu$.

Looking at the form of (32), we see that is closely resembles (14) from Section 2.6. In fact, using the results from Section 2.6, we can write (32) instead as

$$\frac{\alpha}{\pi}\int_{\mu}^{|q|}\frac{dk}{k}\mathcal{I}(\boldsymbol{v},\boldsymbol{v'}), \tag{33}$$

where $\alpha = e^2/(4\pi)$ is the fine structure constant in natural units. Since $\mathcal{I}(\boldsymbol{v},\boldsymbol{v'})$ has no $k$ dependence, the evaluation of the integral is trivial, and we find that the correction to the cross-section is

$$\frac{\alpha}{\pi}\log\left(\frac{-q^2}{\mu^2}\right)\log\left(\frac{-q^2}{m^2}\right). \tag{34}$$

The pair of logs form the *Sudakov double logs*, as mentioned first in Section 1.1. We also see that this correction is $\mathcal{O}(\alpha)$, as would be expected from a single vertex correction to a QED process.

We now ask whether our result would be different for an annihilation process. Using the variable change outlined in Section 2.3, $p \to p, p' \to -p'$, we investigate whether the sign change will modify our result, (34). Looking at (32), the only term containing $p'$ is a fraction with a single $p'$ each in the numerator and denominator. Therefore, under $p' \to -p'$, this term is unchanged, and so (34) is invariant under a crossing operation. This is an unsurprising result, as a different result would raise questions as to whether there was information embedded at the vertex pertaining to whether an annihilation or scattering event were to occur[3].

## 3.2 Vertex Loops

Now that we have derived the Sudakov double logs in the QED case for soft Bremsstrahlung, we will look at the form of the cross-section when there is a loop at the vertex due to photon emission and absorption.

It is possible to derive a function $\Gamma^\mu$ which acts as a vertex coupling and theoretically contains all information about amputated loop corrections to all orders. For our investigation, we evaluate this to first order, effectively the process as given in Figure 3.

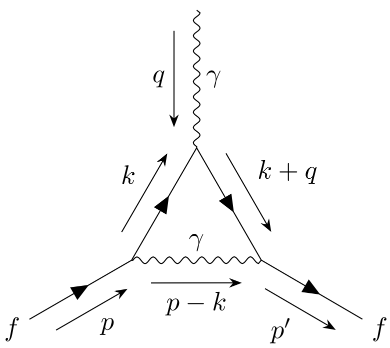


Figure 3: The scattering of a fermion, $f$, off an external source, e.g. another fermion. $f$ enters with momentum $p$, spontaneously emits a photon of momentum $p-k$, which it reabsorbs following the hard scattering event, leaving the diagram with momentum $p'$.

To first order, we may write the vertex function as $\Gamma^\mu = \gamma^\mu + \delta\Gamma^\mu$, where it is clear that $\gamma^\mu$ must be the leading order term without any loops, following from the normal Feynman rules in QED. Since $\gamma^\mu$ requires no simplification, we spend the rest of this subsection evaluating $\delta\Gamma^\mu$, anticipating the recovery of Sudakov logs.

$\overline{u}(p')\delta\Gamma^\mu u(p)$ is scattering amplitude for the 1-loop QED correction to the vertex and can be written as

$$\begin{aligned}&\int \frac{d^4k}{(2\pi)^4}\frac{-ig_{\nu\rho}}{(k-p)^2+i\epsilon}\overline{u}(p')(-ie\gamma^\nu)\\ &\times\frac{i(\not{k}'+m)}{k'^2-m^2+i\epsilon}\gamma^\mu\frac{i(\not{k}+m}{k^2-m^2+i\epsilon}(-ie\gamma^\rho)u(p)\end{aligned} \tag{35}$$

[2] Textbooks such as *An Introduction to Quantum Field Theory* M. Peskin, D. Schroeder [1], or *Quantum Field Theory* F. Mandl, G. Shaw [7].

[3] The vertex at which the soft photon was radiated.

by inspecting Figure 3 and using the Feynman rules (Appendix A), excluding the photon propagator for the external photon. Using the commutation relations and the Dirac equation, we simplify the numerator to the following form

$$(2\pi)^4\overline{u}(p')\Big[\not{k}'\gamma^\mu\not{k} + m^2\gamma^\mu - 2m(k+k')^\mu\Big]u(p). \quad (36)$$

Before further simplifications to the numerator are made, we make simplifications to the denominator which influences our operations on the numerator in future steps. We use the Feynman parameters, as outlined in Section 2.5, to convert the denominator to a polynomial (quadratic) in $k$. We let $A_1 = ((k-p)^2 + i\epsilon), A_2 = (k'^2 - m^2 + i\epsilon)$, $A_3 = (k^2 - m^2 + i\epsilon)$, and use (10) to find replace the propagator denominators. The resulting denominator, $D$, is

$$x(k^2-m^2)+y(k'^2-m^2)+z(k-p)^2+(x+y+z)i\epsilon \quad (37)$$

which may be simplified by simple expansion, substitution of $k' = k+q$, and factorised into $D = k^2 + 2k(yq - zp) + yq^2 + zp^2 - (x+y)m + i\epsilon$. Setting $l = k + yq - zp$, we can complete the square for $l$ in $D$,

$$\begin{aligned} D &= l^2 - xyq^2 + (1-z)^2m^2 + i\epsilon \\ &= l^2 - \Delta + i\epsilon, \end{aligned} \quad (38)$$

and defining $\Delta = -xyq^2 + (1-z)^2m^2$. We now have a simple form for $D$, where $\Delta$ can be thought of as an effective mass term since this is the form of a massive propagator with momentum $l$. Using $l = k+yq-zp$ we perform a change of variables on the numerator, (36). Simple differentiation finds $d^4l = d^4k$, and since $D$ depends only on the magnitude of $l$, the integral can be made spherically symmetric if the numerator depends on $l^2$. The numerator is simplified further by noting that terms within the integrand odd in $l^\mu/D^3$ will integrate to zero, and thus can be removed from the numerator. Additionally, terms containing $l^\mu l^\nu$ factors can be simplified, setting them to

$$l^\mu l^\nu \to \frac{g^{\mu\nu}l^2}{4}. \quad (39)$$

This is possible for the following reasons: (1) since the integral vanishes unless $\mu = \nu$, so the integrand is proportional to a symmetric tensor; (2) since this quantity must also be Lorentz invariant, the only possible tensor satisfying these properties is the metric tensor, so we recover (39). The factor of $1/4$ comes from the dimensionality.

The Mathemetica package *FeynCalc* is employed to apply these changes and simplify the extensive Dirac algebra [12, 13, 14, 15, 16]. Applying the two aforementioned rules to the output of FeynCalc, we find the numerator to be

$$\begin{aligned} \overline{u}(p')\Big[&-\frac{1}{2}\gamma^\mu l^2 + (-y\not{q} + z\not{p}) \\ &\times\gamma^\mu\Big((1-y)\not{q} + z\not{p}\Big) + m^2\gamma^\mu \\ &- 2m\big((1-2y)q^\mu + 2zp^\mu\big)\Big]u(p). \end{aligned} \quad (40)$$

We now decompose this into the form $F_1(q^2)\gamma^\mu + F_2(q^2)i\sigma^{\mu\nu}q_\nu/2m$, where $F_1(q^2), F_2(q^2)$ are form factors of interest. This is done by first decomposing the numerator into the form $A\gamma^\mu + B(p'+p)^\mu + Cq^\mu$ then applying the Gordon Identity, $\gamma^\mu \to (p'+p)^\mu/2m + \sigma^{\mu\nu}q_\nu/2m$, to remove the $(p'+p)^\mu$ terms.

FeynCalc is once again employed to perform this routine with use of the `GordonSimplify[]` function. After this decomposition, we apply the Ward identity, $q_\mu\Gamma^\mu = 0$, to remove all terms proportional to $q_\mu$. Now, putting this numerator back into the integral, with the new denominator, we have a new expression for the first order vertex corrected amplitude

$$\begin{aligned} &2ie^2\int\frac{d^4l}{(2\pi)^4}\int_0^1 dxdydz\delta(x+y+z-1)\frac{2}{D^3} \\ &\times\overline{u}(p')\Big[\gamma^\mu(-\frac{1}{2}l^2 + (1-x)(1-y)q^2 \\ &+ (1-4z+z^2)m^2) \\ &+ \frac{i\sigma^{\mu\nu}q_\nu}{2m}(2m^2z(1-z))\Big]u(p), \end{aligned} \quad (41)$$

We perform a second change of variables on this integral to convert the space from Minkowski to Euclidean, allowing for simpler spherical integration. We let $l^0 = il^0_E$ and $\boldsymbol{l} = \boldsymbol{l_E}$, where the $E$ stands for *Euclidean*, and thus $l_E$ is a *Euclidean 4-momentum*. This is a Wick rotation in the $l^0$ plane. Since the $l^0$ is generally complex, we deform the contour of integration such that it spans the imaginary axis $(-i\infty, i\infty)$ instead of the real axis $(-\infty, \infty)$. Since the contour is continuous and the start/end points cannot change, there are arcs at a radial distance of $\infty$, but these do not contribute to the integral since the integrand quickly drops off at large $|l^0|$. We now use (12) and (13) to evaluate the integral. However, (13) is UV divergent for $m = 3$ and so we introduce a regulator to keep the integral well behaved. We use *Pauli-Villars regularisation*, that is introducing a fictitious large mass $\Lambda$ to regulate the divergence $\Delta \to \Delta + z\Lambda = \Delta_\Lambda$.

The divergent integral is then instead,

$$\begin{aligned} &\int\frac{d^4l}{(2\pi)^4}\left(\frac{l^2}{(l^2-\Delta)^3} - \frac{l^2}{(l^2-\Delta_\Lambda)^3}\right) \\ &= \frac{i}{(4\pi)^2}\int_0^\infty dl_E^2\left(\frac{l_E^4}{(l_E^4-\Delta)^3} - \frac{l^2}{(l^2-\Delta_\Lambda)^3}\right) \\ &= \frac{i}{(4\pi)^2}\log\Big(\frac{\Delta_\Lambda}{\Delta}\Big). \end{aligned} \quad (42)$$

Putting these terms together, the full expression for the amplitude becomes

$$\frac{\alpha}{2\pi}\int_0^1 dxdydz\delta(x+y+z-1)\overline{u}(p')\Big(\gamma^\mu \times\Big[\log\frac{z\Lambda^2}{\Delta}+\frac{1}{\Delta}\Big((1-x)(1-y)q^2(1-4z + z^2)m^2\Big)\Big]+\frac{i\sigma^{\mu\nu}q_\nu}{2m}\Big[\frac{1}{\Delta}2m^2z(1-z)\Big]\Big)u(p). \tag{43}$$

Looking at the individual form factors, we notice that $F_1(0)\neq 1$. At zero momentum transfer, $F_1(0)$ is equal to the charge of the particle, in this case the electron and not what we have derived so far. To restore such physicality to $F_1(q^2)$, we normalise $F_1(q^2)$ by making the substitution $\delta F_1(q^2)\to\delta F_1(q^2)-\delta F_1(0)$, where $\delta F_1(q^2)$ is the first order correction to $F_1(q^2)$.

There is also one more IR divergence arising from the $1/\Delta$ term in $F_1(q^2)$, which must be addressed. We do so by introducing a second mass regulator, $\mu$, which is small, $\Delta\to\Delta+z\mu^2=\Delta_\mu$.

Thus, the form factors can be written,

$$F_1(q^2)=1+\frac{\alpha}{2\pi}\int_0^1 dxdydz\delta(x+y+z-1)\times\Bigg[\log\Big(\frac{m^2(1-z)^2}{m^2(1-z^2)-q^2xy}\Big)+\frac{m^2(1-4z+z^2)+q^2(1-x)(1-y)}{m^2(1-z)^2-q^2xy+\mu^2z}-\frac{m^2(1-4z+z^2)}{m^2(1-z)^2+\mu^2z}\Bigg], \tag{44}$$

$$F_2(q^2)=\frac{\alpha}{2\pi}\int_0^1 dxdydz\delta(x+y+z-1)\Bigg[\frac{2m^2z(1-z)}{m^2(1-z)^2-q^2xy}\Bigg]. \tag{45}$$

We see that $F_1(q^2)$ is IR divergent as $\mu\to 0$, as just discussed, and $F_2(q^2)$ has no divergences[4]. We set $x,y=0$ to study just the IR effects in $F_1(q^2)$. This simplifies the integral,

$$F_1(q^2)\approx 1+\frac{\alpha}{2\pi}\int_0^1 dxdydz\delta(x+y+z-1)\times\Bigg[\frac{-2m^2+q^2}{m^2(1-z)^2-q^2xy+\mu^2}-\frac{-2m^2}{m^2(1-z)^2+\mu^2}\Bigg]. \tag{46}$$

One integral is evaluated using the delta function, and the other two can be handled using a change of variables. We choose $y=(1-z)\xi$ and $w=(1-z)$, thus the integral becomes

$$F_1(q^2)\approx 1+\frac{\alpha}{2\pi}\int_0^1 d\xi\frac{1}{2}\int_0^1 d(w^2)\times\Bigg[\frac{-2m^2+q^2}{w^2(m^2-q^2\xi(1-\xi))+\mu^2}-\frac{-2m^2}{m^2w^2+\mu^2}\Bigg]. \tag{47}$$

Evaluation of the integral over $w^2$ gives

$$F_1(q^2)\approx 1+\frac{\alpha}{2\pi}\int_0^1 d\xi\Bigg[\frac{-2m^2+q^2}{m^2-q^2\xi(1-\xi)}\times\log\Big(\frac{m^2-q^2\xi(1-\xi)}{\mu^2}\Big)+2\log\Big(\frac{-q^2}{\mu^2}\Big)\Bigg]. \tag{48}$$

We have let $m^2\to -q^2$ in the numerator of the logarithmic term since in the limit $\mu\to 0$ it makes little difference, and we only require that the numerator is proportional to either $m^2$ or $-q^2$. We can simplify this expression by collecting the IR divergent term into a single function $f_{\text{IR}}(q^2)$ given by

$$f_{\text{IR}}(q^2)=\int_0^1\Big(\frac{m^2-q^2/2}{m^2-q^2\xi(1-\xi)}\Big)d\xi-1, \tag{49}$$

reducing $F_1(q^2)$ to the form,

$$F_1(q^2)=1-\frac{\alpha}{2\pi}f_{\text{IR}}(q^2)\log\Big(\frac{-q^2}{\mu^2}\Big), \tag{50}$$

Evaluation of $f_{\text{IR}}(q^2)$ in the limit of large $-q^2$ gives

$$f_{\text{IR}}(q^2)\approx\log\Big(\frac{-q^2}{m^2}\Big), \tag{51}$$

and so in this limit, $F_1(q^2)$ is given by

$$1-\frac{\alpha}{2\pi}\log\Big(\frac{-q^2}{m^2}\Big)\log\Big(\frac{-q^2}{\mu^2}\Big). \tag{52}$$

However the correction to the cross-section is proportional to $|\mathcal{M}^2|$, so we square (52) and delete all terms $\mathcal{O}(\alpha^2)$, giving the correction,

$$1-\frac{\alpha}{\pi}\log\Big(\frac{-q^2}{m^2}\Big)\log\Big(\frac{-q^2}{\mu^2}\Big), \tag{53}$$

and so we recover the Sudakov double logs from the consideration of a radiative loop at the electron vertex.

### 3.3 Cancellation of Sudakov Logs

In this next subsection, we will look at the similarities between (34) and (53), and discuss the IR divergences.

Looking at (34) and (53), one can spot the striking similarity between the two equations, that they both contain identical factors. This similarity points to the consideration of *inclusive cross-sections*, first mentioned in Section 1.1. Taking the

[4]Direct evaluation of $F_2(0)$ gives $\frac{\alpha}{2\pi}$, which corrects the g-factor of the electron. An important result in particle physics.

sum of (34) and (53) gives unity. This result suggests that *individually* soft Bremsstrahlung events and vertex loop events are entirely unphysical, since their individual corrections are $\mu$ dependent, on a mass regulator. In principle $\mu = 0$ for a photon, so the individual cross-sections, although having a well defined first order correction, are still IR divergent. However, it would mean to suggest that, since the sum of both corrections has no $\mu$ dependence at all, the total cross-section of these two events together is a physical quantity. Thus, when one considers soft Bremsstrahlung, one *must* consider vertex loops. One could conceptually think of this like they are an inseparable superposition. Individually, we are led to believe that they are each physical processes since they can be constructed legally using the Feynman rules. However, given that it is impossible to reason that a cross-section can *actually* be infinite, we must find a way to unify the two events into a single process which is physical, e.g. that they are two parts to one inseparable process, perhaps. We are then moved to ask whether the Feynman rules are fit for purpose if they permit the construction of unphysical diagrams, and so we question if further research ought to be conducted to place constraints on the construction of Feynman diagrams. Or, maybe, as a physics community, we appreciate that the loose rules allow us to construct more than that which is physical, and so the Feynman rules ought not to be meddled with.

We are inclined to find an intuitive way to understand the Sudakov log cancellation, following from our statement about the (un)physicality of the soft Bremsstrahlung and vertex loop events, i.e. why is each event unphysical on its own, and where does the cancellation come from? In either case (soft Bremsstrahlung or vertex loops) a photon is emitted, so we could speculate that in the event a photon is radiated, we must consider the sum of all possible effects of this emission (subsequent absorption, pair production, etc.), giving a result which cancels the Sudakov logs. Since this is generally true, especially when considering event generation, this thinking does not give us any more insight than quantum mechanics. Alternatively, as our method has been to $1^{\text{st}}$ order, it could be that we only have to consider the whole $1^{\text{st}}$ order subset of the infinite possible radiative perturbations, and likewise in a $2^{\text{nd}}$ order treatment, we must consider all possible $2^{\text{nd}}$ order processes.

However, this would still not reduce the cancellation down to its simplest structure. The set of $1^{\text{st}}$ order radiative corrections is actually still a superset of the set required to cancel the Sudakov logs. We have not had to consider the exchange of a photon *between the electron and the muon* to provide complete cancellation of the IR divergences, a 'box diagram', and as such the set of $1^{\text{st}}$ order radiative corrections would over specify the processes needed to provide a cancellation of the Sudakov logs. This does raise further questions as to why we have not had to consider box diagram style exchanges, especially since photon exchange between different particle/antiparticle species in $e^-\mu^- \to e^-\mu^-$ is necessary for a full $1^{\text{st}}$ order calculation of the amplitude. However, exercising this line of inquiry does tell us that the QED Sudakov logs must be specifically due to photon emission and exchange on a single particle/antiparticle species, e.g. $e^+, e^-$. However, we cannot specialise this conceptual understanding of Sudakov logs any further. For instance, it would be impossible to pin the Sudakov log cancellation down to a specific/identifiable particle. This is because its quantum numbers are different before and after the hard event (due to a momentum change), thus we cannot identify a single particle as the same before or after an event. Additionally, due to crossing symmetry, the cancellation of Sudakov logs must occur for an annihilation event, where the vertex loop is over an electron and positron, two distinct particles at the same time. Thus, again, it would be impossible to pin the cancellation down to the effect of a single particle.

These and other philosophical insights into the physical understanding of the cancellation of Sudakov logs were largely conducted by myself.

### 3.4 Observables

We will end this section by deriving some observable quantities using the idea of an inclusive cross-section.

The sum of the corrections due to soft Bremsstrahlung and vertex loops exclusive of the large $-q^2$ limit is given by the sum of (33) and (50). To derive a useful cross-section, we are motivated to prove that $\mathcal{I}(\boldsymbol{v}, \boldsymbol{v'}) = 2f_{\text{IR}}(q^2)$ for arbitrary $q^2$, since this will provide the same cancellation in the non large $-q^2$ limit. The *measured cross-section* of the scattering event will be the sum of the cross-sections where scattering is observed, but a photon is not. The observed cross-section is, therefore, the cross-section that leading-order scattering occurs multiplied by two corrections: (1) a virtual photon is exchanged between the incoming and outgoing electron (fermion); (2) a real photon is emitted but is below the detector sensitivity.

Trivially performing the integral in (33) over $k$ but up to the energy $E_l$ not $-q^2$ gives the IR divergent correction that a photon is emitted but not detected. This is given by

$$\frac{\alpha}{2\pi}\log\left(\frac{E_l^2}{\mu^2}\right)\mathcal{I}(\boldsymbol{v}, \boldsymbol{v'}). \tag{54}$$

Looking at (15), we have

$$\begin{aligned}\mathcal{I}(\boldsymbol{v}, \boldsymbol{v'}) = \int \frac{d\Omega_{\boldsymbol{k}}}{4\pi}\bigg(&\frac{2p\cdot p'}{(k\cdot p')(k\cdot p)} - \frac{m^2}{(k\cdot p')} \\ &- \frac{m^2}{(k\cdot p)}\bigg),\end{aligned} \tag{55}$$

where the last two terms integrate to $1/m^2$ since

$$\int \frac{d\Omega_{\boldsymbol{k}}}{4\pi}\frac{1}{(\hat{k}\cdot p)^2} = \frac{1}{2}\int_{-1}^{1}\frac{d\cos\theta}{(p^0 - p\cos\theta)^2} = \frac{1}{m^2}. \tag{56}$$

We have used here the usual 'kinematic' style decomposition of interpreting the dot product with an angle between vectors. The first term in the integral is evaluated using Feynman parameters. That is

$$\begin{aligned}
&\int \frac{d\Omega_{\boldsymbol{k}}}{4\pi} \frac{1}{(\hat{k} \cdot p)(\hat{k} \cdot p')} \\
&= \int_0^1 d\xi \int \frac{d\Omega_{\boldsymbol{k}}}{4\pi} \frac{1}{[\xi \hat{k} \cdot p' + (1-\xi)\hat{k} \cdot p]^2} \\
&= \int_0^1 \frac{d\xi}{[\xi p' + (1-\xi)p]^2} = \int \frac{d\xi}{m^2 - \xi(1-\xi)q^2},
\end{aligned} \tag{57}$$

where $q^2 = 2m^2 - 2p \cdot p'$.

Thus we can express the differential intensity as

$$\mathcal{I}(\boldsymbol{v}, \boldsymbol{v'}) = \int_0^1 \left( \frac{2m^2 - q^2}{m^2 - \xi(1-\xi)q^2} \right) d\xi - 2, \tag{58}$$

which upon comparison with (49) immediately proves

$$\mathcal{I}(\boldsymbol{v}, \boldsymbol{v'}) = 2 f_{\text{IR}}(q^2). \tag{59}$$

Thus, taking the sum of the corrections for the measured cross-section, (50) and (54), and substituting (59) we get

$$1 - \frac{\alpha}{\pi} f_{\text{IR}}(q^2) \log\left( \frac{-q^2}{m^2} \right), \tag{60}$$

which has no $\mu$ dependence, so is finite, and therefore observable. Experimental use requires more care in handling (49), however we can use the large $-q^2$ limit to recover a Sudakov double log style correction,

$$1 - \frac{\alpha}{\pi} \log\left( \frac{-q^2}{m^2} \right) \log\left( \frac{-q^2}{E_l^2} \right). \tag{61}$$

For example, the cross-section for the scattering of an electron and muon to first order is

$$d\sigma = d\sigma_0 \left[ 1 - \frac{\alpha}{\pi} \log\left( \frac{-q^2}{m^2} \right) \log\left( \frac{-q^2}{E_l^2} \right) \right], \tag{62}$$

where $d\sigma_0$ is the infinitesimal cross-section of the leading order process.

# 4 Radiative Corrections in Electroweak Physics

Motivated by the cancellation of Sudakov double logarithms in the QED case, we now study the radiative corrections in the electroweak theory. In this section, we re-derive scalar corrections to the leading order cross-section for soft Bremsstrahlung using a method analogous to that previously detailed in Section 3.1. Furthermore, we attempt to recover similar vertex loop corrections in the electroweak theory to that previously derived for QED in Section 3.2. Since the electroweak theory introduces two new bosons in addition to the pre-existing photon, many more diagrams contribute to a single scattering or annihilation process in the electroweak theory than in QED. The additional complexity inspires us to search for a general method for solving electroweak vertex problems. We then conclude the discussion of electroweak radiative corrections by re-exploring IR divergences and the cancellation of Sudakov logarithms.

## 4.1 Soft Bremsstrahlung

We now wish to find a correction analogous to (34) in the electroweak theory. For this subsection, we will focus on a case similar to the soft Bremsstrahlung seen in Figure 2, however, the radiated particle is rather a $Z^0$-boson instead of a photon. Our method follows similarly from Section 3.1, starting with the construction of the scattering amplitude, which is given by

$$\begin{aligned}
&\overline{u}(p') \Bigg( \mathcal{M}_0(p', p-k) \frac{i(\not{p} - \not{k} + m)}{(p-k)^2 - m^2} \\
&\times \left( -\frac{ig_z}{2} \gamma^\mu (v_f - a_f \gamma^5) V_{ij} \right) \epsilon^*_\mu(k) \\
&+ \left( -\frac{ig_z}{2} \gamma^\mu (v_f - a_f \gamma^5) V_{ij} \right) \epsilon^*_\mu(k) \\
&\times \frac{i(\not{p}' - \not{k} + m)}{(p'-k)^2 - m^2} \mathcal{M}_0(p'+k, p) \Bigg) u(p),
\end{aligned} \tag{63}$$

where we have used the Feynman rules in the electroweak theory to substitute the vertex couplings from (26) to the relevant $Z^0$-boson couplings, $-ig_z\gamma^\mu(v_f - a_f\gamma^5)V_{ij}/2$. It should be noted that these couplings are distinctly different from the QED photon coupling, $-ie\gamma^\mu$, due to the $\gamma^5$ term. The $\gamma^5$ acts to select only the left-handed particles, the only such particles that couple in electroweak physics. Thus, there are two additional important coupling constants in the electroweak theory, as opposed to QED. These are the $v_f$ (vector) and $a_f$ (axial vector) couplings seen in (63), which distinguish the strengths of the coupling to chiral symmetric processes ($v_f$) and processes which differ between left and right handed particles ($a_f$). The $v_f$ and $a_f$ terms have different expressions for different particles. Although we intend to study just the scattering event $e^-\mu^- \to e^-\mu^-$, we will leave the couplings in a general form, since our treatment of the scattering amplitude is quite general.

We next make some simplifying assumptions for (63), such as assuming the photon momentum $k$ is soft. This has the same effect as in Section 3.1 where the amplitude for the hard event, $\mathcal{M}_0$, is approximately the same whether the emission occurs before or after the hard process. Additionally, we set $V_{ij} = 1$, since we intend to work only with leptons. Finally, since we only care about the high energy limit,

$s \gg m_Z^2$, we will work in the limit of massless fermions. In such a case, we need not worry about any coupling to the Higgs sector and can make similar simplifications to the denominator of the amplitude using the Mandelstam invariants (or equivalently taking the massless limit), as laid out in Section 2.4 and 3.1.

Exploiting these simplifications and applying them to the amplitude, we find the amplitude, $i\mathcal{M}$, becomes

$$\overline{u}(p')\Bigg(\mathcal{M}_0(p',p)\frac{-(\not{p}+m)}{2p\cdot k}\Big(\frac{g_z}{2}\gamma^\mu(v_f-a_f\gamma^5)\Big)\epsilon_\mu^*(k) + \Big(\frac{g_z}{2}\gamma^\mu(v_f-a_f\gamma^5)\Big)\epsilon_\mu^*(k)\frac{(\not{p}'+m)}{2p'\cdot k}\mathcal{M}_0(p',p)\Bigg)u(p). \tag{64}$$

Next, we focus on simplifying the numerator of (64), with the aim of collecting $\gamma^\mu$ and $\gamma^\mu\gamma^5$ terms separately, and at the same time moving them all to the left. Where possible, we try to convert terms to scalars either by using the Dirac equation or multiplying vectors with other vectors to produce an inner product. The numerator in the top line of (64) is simplified to

$$\begin{aligned}&\frac{g_z}{2}\Big(v_f(\not{p}+m)\gamma^\mu\epsilon_\mu^* - a_f(\not{p}+m)\gamma^\mu\gamma^5\Big)u(p)\\ &= g_z\Big(v_f-a_f\gamma^5\Big)p\cdot\epsilon^* u(p).\end{aligned} \tag{65}$$

The explicit details of the algebra are given in Appendix C for brevity. The other numerator term is simplified similarly to

$$\overline{u}(p')g_z(v_f+a_f\gamma^5)p'\cdot\epsilon^*, \tag{66}$$

where it is clear the only difference between these terms, and therefore the difference between the numerators for the soft Bremsstrahlung before or after the hard process is a sign change on $a_f$, and different momenta, naturally. Inserting (65) and (66) into (64), we find that the amplitude becomes

$$\frac{g_z}{2}\Bigg(\overline{u}(p')\mathcal{M}_0(v_f-a_f\gamma^5)u(p)\Bigg)\Big[\frac{p'\cdot\epsilon^*}{p'\cdot k}-\frac{p\cdot\epsilon^*}{p\cdot k}\Big], \tag{67}$$

after expanding the terms containing the coupling constants, and factorising all scalar terms.

(67) is an important result, and a milestone in the derivation of the corrections in electroweak theory. Already, looking at (67), we see that there is a component proportional to $\overline{u}(p')\mathcal{M}_0u(p)$ multiplied by a scale factor involving the electroweak coupling constant for $Z^0$-bosons, $g_z$, the vector coupling, $v_f$, the particle momenta, $p, p'$, and the photon polarisation vector, $\epsilon^*$. The term in square brackets is identical to the correction in (30), suggesting that this term is related to the behaviour of soft Bremsstrahlung, independent of the framework, e.g. QED, EW theory. The term proportional to $\overline{u}(p')\mathcal{M}_0u(p)$ is identical to the QED case up to the scale factor, and so can be given the same treatment as the derivation in Section 4.1. However, there is a term proportional to $\overline{u}(p')\mathcal{M}_0\gamma^5u(p)$, which requires additional handling.

The next step requires squaring the amplitude to derive the correction to the cross-section. However, since the current expression in-between the spinors is quite different to the QED case, we cannot mimic the QED method verbatim. Instead, we explicitly square the amplitude in the hope that this reduces to simpler expressions similar to the QED case. Since $\mathcal{M}$ is generally complex, squaring means multiplication by the complex conjugate, $\mathcal{M}^*$, which is equal to the Hermitian conjugate in this case, $\mathcal{M}^\dagger$. Finding the Hermitian conjugate will require an assumption on $\mathcal{M}_0$, in addition to some new definitions which we detail now.

We assume that $\mathcal{M}_0$ contains an odd number of $\gamma$-matrices, and so $\{\mathcal{M}_0,\gamma^5\}=0$, from $\{\gamma^\mu,\gamma^5\}=0$. We also define the notation $\overline{\mathcal{M}_0}\equiv\gamma^0\mathcal{M}_0^\dagger\gamma^0$, which gives an easy way to take the Hermitian conjugate of an amplitude containing $\mathcal{M}_0$ in any representation. Applying this to the term proportional to $v_f$, we trivially derive its Hermitian conjugate

$$\Big(\overline{u}(p')\mathcal{M}_0v_fu(p)\Big)^\dagger = \overline{u}(p)v_f\overline{\mathcal{M}}_0u(p'). \tag{68}$$

Elements containing $\mathcal{M}_0$ and $\gamma^5$ require slightly more algebra when taking the Hermitian conjugate, but the overall result yields a simple minus sign compared to the case without any $\gamma^5$ terms,

$$\Big(\overline{u}(p')\mathcal{M}_0a_f\gamma^5u(p)\Big)^\dagger = -\overline{u}(p)\gamma^5a_f\overline{\mathcal{M}}_0u(p'). \tag{69}$$

Wishing to find a correction to the cross-section, we now square (67) using the results just derived. The cross-terms do not contribute to the overall cross-section, so we only keep terms proportional to $\overline{u}(p')\mathcal{M}_0u(p)\overline{u}(p)\overline{\mathcal{M}}_0u(p')$ and $\overline{u}(p')\mathcal{M}_0u(p)\gamma^5\overline{u}(p)\overline{\mathcal{M}}_0\gamma^5u(p')$. This gives an expression for the squared cross-section,

$$\begin{aligned}|\mathcal{M}|^2 &= \frac{g_z^2}{4}\Big[\frac{p'\cdot\epsilon^*}{p'\cdot k}-\frac{p\cdot\epsilon^*}{p\cdot k}\Big]^2\\ &\times\{\overline{u}(p')\mathcal{M}_0v_fu(p)\overline{u}(p)\overline{\mathcal{M}}_0v_fu(p')\\ &+\overline{u}(p')\mathcal{M}_0a_f\gamma^5u(p)\overline{u}(p)\overline{\mathcal{M}}_0a_f\gamma^5u(p')\}.\end{aligned} \tag{70}$$

We now use trace algebra to simplify the terms in (70), since $\mathrm{Tr}(|\mathcal{M}|^2)=|\mathcal{M}|^2$, as $|\mathcal{M}|^2$ is scalar. It is possible to exploit the symmetry of the cyclic permutations of the trace to collect spinors and anti-spinors and thus make use of the completeness relations.

The first term is trivially,

$$\begin{aligned}&\mathrm{Tr}(\overline{u}(p')\mathcal{M}_0v_fu(p)\overline{u}(p)\mathcal{M}_0v_fu(p'))\\ &= v_f^2|\overline{u}(p')\mathcal{M}_0u(p)|^2,\end{aligned} \tag{71}$$

which mimics the QED case, as expected. The second term is calculated similarly, taking care with the $\gamma^5$'s,

$$\begin{aligned}&\mathrm{Tr}(\overline{u}(p')\mathcal{M}_0 a_f\gamma^5 u(p)\overline{u}(p)\mathcal{M}_0 a_f\gamma^5 u(p'))\\&= a_f^2|\overline{u}(p')\mathcal{M}_0 u(p)|^2.\end{aligned} \tag{72}$$

Inserting (71) and (72) into (70), we find the square amplitude becomes

$$\frac{g_z^2}{4}\left[\frac{p'\cdot\epsilon^*}{p'\cdot k}-\frac{p\cdot\epsilon^*}{p\cdot k}\right]^2|\overline{u}(p')\mathcal{M}_0 u(p)|^2(v_f^2+a_f^2), \tag{73}$$

and thus we have a correction to the leading order scattering process. The correction is given by

$$\frac{g_z^2}{4}(v_f^2+a_f^2)\left[\frac{p'\cdot\epsilon^*}{p'\cdot k}-\frac{p\cdot\epsilon^*}{p\cdot k}\right]^2, \tag{74}$$

an analogous result to the square of (31). Once again, we can check with crossing symmetry whether this changes for an annihilation event, and indeed it does not. Similarly, this is expected for reasons previously foretold (see Section 4.1).

Now, we must integrate over the phase-space, but since we see that the square brackets term in (74) is equal to that seen previously, namely in (32) and therefore (14), we can jump to the total correction by comparing (74) with (33), giving

$$\frac{g_z^2}{32\pi^2}(v_f^2+a_f^2)\log\left(\frac{-q^2}{m_Z^2}\right)\mathcal{I}(\boldsymbol{v},\boldsymbol{v'}), \tag{75}$$

where the log term is cut-off by the non-fictitious $Z^0$-boson mass, $m_Z$, since the $Z^0$-boson propagator contains no IR divergence. Similarly, in the large $-q^2$ limit, $\mathcal{I}(\boldsymbol{v},\boldsymbol{v'})$ is cut-off by the $m_Z$ mass instead of the regulator $\mu$. Therefore in the large $-q^2$ limit, we find the correction to the cross-section to be

$$\frac{g_z^2}{16\pi^2}(v_f^2+a_f^2)\log^2\left(\frac{-q^2}{m_Z^2}\right), \tag{76}$$

and so we have recovered the Sudakov double logs in the electroweak theory for the case of soft $Z^0$-boson Bremsstrahlung. This result agrees with the literature, that cross-sections involving $Z^0$-boson exchange contain corrections proportional to (76) [17].

Interestingly, although the electroweak and QED cases both find corrections proportional to double logs, (76) differs fundamentally from the QED case, (34) because the electroweak correction is finite due to the massive boson propagator regulated naturally by $m_Z$.

## 4.2 Vertex Loops

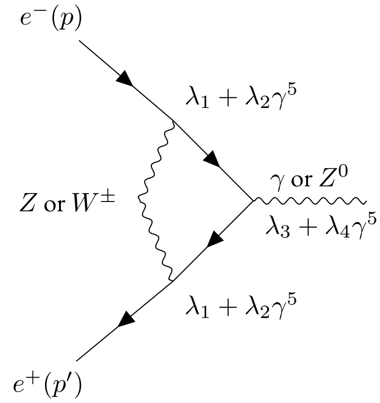


Figure 4: The general 1st order vertex correction for an annihilation process. The $\lambda_i$ coefficients denote the individual couplings for vector and axial-vector components of the vertices. The $\lambda_4$ component is zero in this diagram due to the explicit condition that a photon mediates the hard process.

In this penultimate section, we devote our inquiry to the study of general vertex corrections in the electroweak theory. Since there are many vertex corrections that can be derived for different radiative emission ($\gamma$, $Z^0$, $W^\pm$) and boson exchange in the hard process ($Z^0$, $\gamma$), it is most efficient to consider the most general vertex in the annihilation process, Figure 4. The method in this section follows from the work in [9, 10] as detailed in Section 2.7 and was conducted largely by my collaborator.

The most general vertex coupling for a fermion-fermion-vector process is proportional to

$$T^\mu \propto [F_v\gamma^\mu + G_A\gamma^\mu\gamma^5], \tag{77}$$

in the limit of massless fermions, where $g$ is a coupling constant, and $F_v$, $G_A$ are the vector and axial-vector form factors. The form factors can be expressed in the form

$$\begin{aligned}F_v &= \frac{g^2}{16\pi^2}f_v V,\\ G_A &= \frac{g^2}{16\pi^2}g_A V,\end{aligned} \tag{78}$$

where $g$ is a coupling constant. The $V$ component contains information about the integral over the loop, as well as the momenta, and the $f_v$ and $g_A$ terms contain information about the chiral nature at each vertex, which we explain further shortly.

We wish to evaluate these form factors to first order, much like that for QED in Section 4.2. In Figure 2.7, it can be seen that the vertices have been defined generally, such that they depend on $\lambda_i$ parameters, which are the vector and axial-vector couplings for a given vertex. For example, looking at the Feynman rules for QED and electroweak theory (see Appendix A), we see that in the case of a pure QED vertex, there is only one possible mapping, $\lambda_1 = Q, \lambda_2 = 0$, and for the case of a $Z^0$ boson as we considered in the last section, the mapping would be $\lambda_1 = v_f, \lambda_2 = a_f$. It is these lambda factors that make up the $f_v$ and $g_A$ terms, which will be determined shortly.

From Figure 4, and mimicking the method in Section 4.2, we write the scattering amplitude for the general 1-loop electroweak correction. Here we just give the form of the general numerator, since the denominator is naturally processed in the PaVe integral as they are the same, see (21).

The general numerator can be written as

$$\begin{aligned} &g^{\mu\rho}\overline{v}(p)\gamma^{\nu}(\lambda_1 - \lambda_2\gamma^5)\not{k}\gamma^{\mu}(\lambda_3 - \lambda_4\gamma^5) \\ &\times(\not{k} + \not{p} + \not{p}')\gamma^{\rho}(\lambda_1 - \lambda_2)u(p'), \end{aligned} \tag{79}$$

by applying the relevant Feynman rules and ignoring all propagator denominator terms. In FeynCalc, this is then simplified and expanded into a linear combination of $1, k_\mu, \ldots$. The $1, k_\mu, \ldots$ terms are then substituted for the relevant $C$ integrals, and the integral is thus evaluated. The result from FeynCalc in $n$ dimensions, neglecting terms not proportional to $\gamma^\mu, \gamma^\mu\gamma^5$, is

$$\begin{aligned} &[(\lambda_1^2\lambda_3 + 2\lambda_1\lambda_2\lambda_4 + \lambda_2^2\lambda_3)\overline{v}(p)\gamma^{\mu}u(p') \\ &\times(2(n-4)(p'\cdot(C_{11}p' + C_{12}) \\ &+ 4(p'\cdot(C_{11}p + C_{12}p') - (n-2)(C_{21}p_1^2 \\ &+ C_{22}p'^2 - C_{23}\frac{s}{2} + (n-2)C_{24})] \\ &+ [(\lambda_1^2\lambda_4 + 2\lambda_1\lambda_2\lambda_3 + \lambda_2^2\lambda_4)((n-4) \\ &\times(p\cdot(C_{11}p + C_{12}p')) + 2(p'\cdot(C_{11}p \\ &+ C_{12}p')) + (n-2)(C_{23}s - C_{24}(n-2)) \\ &\overline{v}(p')\gamma^{\mu}\gamma^5u(p)]. \end{aligned} \tag{80}$$

Comparing (80) with (77) we see that since (80) is a linear combination of terms proportional to $\gamma^\mu$ and $\gamma^\mu\gamma^5$ we have derived $F_v$ and $G_A$. We now give the $V$, $f_v$ and $g_A$ terms by comparison, neglecting the $p, p'$ terms, and setting $n = 4$,

$$\begin{aligned} f_v &= \lambda_1^2\lambda_3 + 2\lambda_1\lambda_2\lambda_4 + \lambda_2^2\lambda_3, \\ g_A &= \lambda_1^2\lambda_4 + 2\lambda_1\lambda_2\lambda_3 + \lambda_2^2\lambda_4 \end{aligned} \tag{81}$$

and

$$V = 2s(C_{11} + C_{23}) - 4C_{24}, \tag{82}$$

both of which agree with [9].

Now, [11] gives the $C_{ij}$ terms as logarithmic expansion in the large $-q^2$ limit. To first order, the relevant terms are

$$\begin{aligned} C_{11} &= -\frac{1}{2s}\log^2\Big(\frac{-q^2}{m_2^2}\Big) + \frac{1}{s}\log\Big(\frac{-q^2}{m_1m_2}\Big), \\ C_{23} &= \frac{1}{s}\log\Big(\frac{-q^2}{m_2m_3}\Big), \\ C_{24} &= -\frac{1}{4}\log\Big(\frac{-q^2}{\mu^2}\Big). \end{aligned} \tag{83}$$

Thus, using $F_v$ and $G_A$, we can jump straight to the corrections to the cross-section by using the same methods outlined previously. That is, summing the squares of the form factors, recalling that this is a 1st order correction so the correction includes a $1+$ term and the $\mathcal{O}(g^4)$ terms are neglected, so the correction is $1 + 2F_v + 2G_A + \mathcal{O}(g^4)$.

In the case of a vertex loop mediated by a $Z^0$ boson, we get the following infinitesimal cross-section

$$\frac{g_z^2(v_f^2 + a_f^2)}{32\pi^2}\Big[1 - \log^2\Big(\frac{-q^2}{m_Z^2}\Big) + 4\log\Big(\frac{-q^2}{m_Zm_e}\Big)\Big]d\sigma_0, \tag{84}$$

and for a $W^\pm$ boson, we get

$$d\sigma_0 + \frac{g_W^2}{32\pi^2}\Big[-\log^2\Big(\frac{-q^2}{M_W^2}\Big) + 4\log\Big(\frac{-q^2}{M_Wm_\nu}\Big)\Big]d\sigma_1, \tag{85}$$

where $d\sigma_0$ is the leading order infinitesimal cross-section, and $d\sigma_1$ is a component of the infinitesimal cross-section derived from the interference of the $W^\pm$ vertex correction with leading order. The $d\sigma_1$ component is a result of the lack of the ability to factorise the $W^\pm$ vertex from the first-order cross-section. Thus, we end our derivation of the electroweak vertex corrections.

## 4.3 Cancellation of Sudakov Logs

In this culminating section, we now revisit the idea of cancellation of Sudakov double logs by inspecting (76), (84) and (85).

Unfortunately, (84) and (85) do not agree with the literature, [17], since each is incorrect by a factor of $1/2$ in the $\log^2$ term. This means to suggest that there is an error in one or more of the algebraic steps. An aside calculation starting from the results for the general vertex form factors given in [9] (using our conventions), also yields the same corrections to the cross-sections, (84) and (85). In this case, it could be that all the necessary algebraic steps were taken and performed correctly, however, conventions may have been misaligned from one source to another which could have been overlooked.

It was anticipated that the double log corrections for $Z^0$-bosons, (76) and (84), would cancel given that the same is true in QED. However, since there is a factor of $1/2$ difference in the terms proportional to double logs, the cancellation of $Z^0$-boson is not complete.

However, we may speculate on the properties of such a system where the cancellation of the double log terms is complete. Although not directly related to the results of the last subsection, this speculation might allow for insight into whether the factor of $1/2$ is physical or an error, despite disagreeing with the pre-existing literature. Should a compelling case for cancellation of the $Z^0$ contributions arise, then it would amount to suggesting that the factor of $1/2$ is an error in the calculations or a mismatching of conventions.

Supposing that there is complete cancellation of the terms proportional to double logs in (76) and (84), we still find a single log residual term, $4\log(-q^2/m_Z m_e)$. Despite this non-cancellation, we can suggest that a cross-section inclusive of these two processes might cancel, because our treatment of the large $-q^2$ limit of (15), or the approximate splitting of (15) into (19), in Section 2.6 may have unintentionally suppressed single log terms. These terms may not have been neglected in the full calculation of the $C_{ij}$ coefficients in [11], leading to the residual single log term. This also supports the narrative that there may be discordance in the conventions used between our study and [9, 11].

However, even presuming that the cancellation of the $Z^0$ contributions is fully complete, it would still be ill-advised to construct such a cross-section where just these processes are considered since these form only two of the five processes where the hard event is mediated by a photon. In a cross-section inclusive of all five contributions, the photon radiation (Section 3.1) and photon loop (Section 3.2) would cancel as described previously, in addition to the cancellation of the $Z^0$-boson radiation and $Z^0$-loop. However, the fifth contribution, a $W^\pm$ vertex loop, has no radiative (Bremsstrahlung) counterpart due to the flavour change on the lepton which occurs. Explicitly, since the electron changes to a neutrino after $W^\pm$ radiation, no annihilation can occur because the neutrino and positron are not mutual antiparticles. If we expect that the cancellation of double logs is representative of the cancellation of soft Bremsstrahlung and vertex loops, then we would not find cancellation of the Sudakov logs in $e^+e^- \to \mu^+\mu^-$ because of the missing radiative counterpart to the $W^\pm$ vertex loop. As such, in an inclusive cross-section, the $W^\pm$ contribution remains as a residual, and there is a large correctional factor to the cross-section. Furthermore, since the $W^\pm$ is massive, the correction is IR finite, so this correction would be observable.

Although a more detailed calculation, inclusive of say a $Z^0$ mediating the hard process, may yield different results, the cancellation of the $W^\pm$ vertex loop and its radiative partner, cannot be reconciled due to the absence of a case where a $W^\pm$ is emitted *and* annihilation occurs. This is likely indicative of the nature of the electroweak force, that charged and uncharged leptons exist within doublets, though may still be observed unbound from this doublet. Since the electroweak force has no confinement mechanism, unlike QCD which does, we are able to produce a pure beam of given lepton species, an effect of which we observe here with the lack of $W^\pm$ Sudakov double log cancellation. If it were the case that the beam were also populated with equal parts $e^+, e^-$ and $\nu_e, \overline{\nu}_e$, then under the radiative emission of a $W^\pm$ the resulting electron-neutrino *would* be able to annihilate with an anti-electron-neutrino from the beam. It could be possible that the 1st order correction to this scenario does cancel with the vertex loop correction.

This insight provides some more validation of the speculation for the QED cancellation discussed in Section 3.3. There, we suggested that the cancellation of Sudakov logs was due to the emission and absorption of a photon from a single particle/antiparticle species. At the same time, we suggested that it was impossible to specialise this answer further. Here, we have also been unable to make any more specialisations to this conceptualisation. However, we have been able to expand the set of particles/antiparticles for which cancellation occurs to include an entire electroweak particle/antiparticle doublet, in the case that we allow for electroweak interactions.

Looking back at what was discussed in Section 3.3, we can see that the statement that "the [cancellation of] QED Sudakov logs must be specifically due to photon emission and exchange on a single particle/antiparticle species" is equivalent to stating that cancellation can only occur for processes which allow for annihilation. These statements are equivalent since in the QED case, where there is no flavour changing, annihilation can only occur for particles/antiparticles of the same type, e.g. $e^+, e^-$. After we relax the condition that exchange is limited to a single particle/antiparticle species, and instead to a whole leptonic doublet, the $W^\pm$ considerations from earlier compound this thinking. Due to the flavour changing nature of the $W^\pm$, annihilation can still occur within a given leptonic doublet (and its antiparticle counterparts).

Thus, we are able to overwrite our conceptual take on the cancellation of Sudakov logs from Section 3.3 with something much more digestible. If our line of thinking is correct, then Sudakov log cancellation can be conceptually understood by understanding which processes in a given theory allow for annihilation to occur both for soft Bremsstrahlung and for vertex loops. Thus, it would be easy to determine if the double log corrections cancel by constructing a few Feynman diagrams.

In hindsight now, it is possible to use the aforementioned considerations to suggest that the factor of $1/2$ missing from (84) and (85) is in fact an algebraic or conventional error. This is because we have suggested that the cancellation of Sudakov logs is due to matching pairs of soft Bremsstrahlung *and* vertex loop corrections which both allow for annihilation to occur. Since the $Z^0$-boson allows for both these corrections, we would expect them to cancel. If the factor $1/2$ is a true result, then it would be impossible to develop such simple conceptual analyses. Although obviously not near a guaranteed proof that the factor $1/2$ is a mathematical or conceptual error, Occam's razor would suggest that since the simplest explanation is often the correct one, our simple explanation for the cancellation of Sudakov logs suggests that the factor of $1/2$ is probably either mathematical or conventional.

The analysis in this section was conducted largely by myself.

# 5 Conclusions and Further Work

In this report, we told the fable of *the cancellation of Sudakov double logarithms in QED and the electroweak theory*. In the QED case, we proved that the individual consideration of soft emission or vertex loops leads to inconsistencies at the cross-section calculation, as suggested by irremovable IR divergences of the form

$$\frac{\alpha}{\pi}\log\Big(\frac{-q^2}{m^2}\Big)\log\Big(\frac{-q^2}{\mu^2}\Big). \tag{86}$$

We then showed that inclusive consideration of these processes cancels the divergences, a result previously derived by Bloch and Nordsieck [2]. We suggested that since a physical result is only obtained when an inclusive cross-section is studied, the two processes should not be considered as separate in their own right, and rather that they are two inseparable parts of the same process. We concluded our study of radiative corrections in QED by providing a correction to a scattering or annihilation event, inclusive of the $1^{\text{st}}$ order radiative corrections previously calculated. We found this to be

$$d\sigma = d\sigma_0\left[1-\frac{\alpha}{\pi}\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{E_l^2}\right)\right], \tag{87}$$

where $E_l$ is the minimum photon energy that can be detected.

We then explored the same corrections in the electroweak theory, recovering the Sudakov logarithms. For soft Bremsstrahlung, the correction to the form factor was found to be

$$\frac{g_z^2}{16\pi^2}(v_f^2+a_f^2)\log^2\left(\frac{-q^2}{m_Z^2}\right), \tag{88}$$

to $1^{\text{st}}$ order. Following this calculation, we derived the corrections for vertex loops with a $Z^0$ or $W^\pm$ in the loop. For $Z^0$, the correction is

$$\frac{g_z^2(v_f^2+a_f^2)}{32\pi^2}\left[1-\log^2\left(\frac{-q^2}{m_Z^2}\right)+4\log\left(\frac{-q^2}{m_Z m_e}\right)\right]d\sigma_0, \tag{89}$$

and for $W^\pm$ it is

$$d\sigma_0+\frac{g_W^2}{32\pi^2}\left[-\log^2\left(\frac{-q^2}{m_W^2}\right)+4\log\left(\frac{-q^2}{m_W m_\nu}\right)\right]d\sigma_1, \tag{90}$$

where the latter two corrections depart from the literature.

We found that in the $Z^0$ case, there is a partial cancellation of the Sudakov logs. However, with a $W^\pm$ in the loop, there is no cancellation with soft Bremsstrahlung, as no such annihilation event can occur for $W^\pm$ radiative emission when only electrons and positrons are available. Therefore, the $W^\pm$ leaves a residual correction to the cross-section in an inclusive regime. However, due to the $W^\pm$ mass, this correction is naturally regulated, and so there are no IR divergences. This correction should therefore be observable.

For future work, we are interested in the cancellation of these logarithms beyond $1^{\text{st}}$ order and are inspired to study the non-cancellation of the logarithms in the electroweak case with more detail. Additionally, we place intrigue on investigating radiative corrections to an inclusive cross-section of a beam homogeneous in all leptons. Moreover, we place emphasis on the investigation of an intuitive understanding of these processes, and that some future research should be dedicated to producing a more inclusive and physical understanding of these processes and the cancellations of their corrections. Such inquiry is important for public consumption of physics, which ought to be accessible to everyone.

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(a) $\frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}$ (b) $\frac{-i}{p^2+i\varepsilon}\left(g^{\mu\nu}-(1-\xi)\frac{p^\mu p^\nu}{p^2}\right)$

Figure 5: Feynman propagators for fermions (a) and photons (b). In the Feynman gauge, $\xi = 0$.

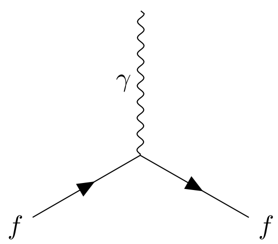


Figure 6: The vertex coupling in QED, given by $iQe\gamma^\mu$. For electrons, $Q = -1$.

# A Feynman Rules

This appendix briefly summarises the Feynman rules relevant to this report.

## A.1 QED

The Feynman rules provide a simple foundation for constructing scattering amplitudes in quantum field theory. The QED rules are taken from [1]. The Feynman rules for propagators in QED are outlined in Figure 5. The QED vertex coupling is detailed in Figure 6. The rules for external lines are that fermions are given by spinors $u$ and $\overline{u}$ for incoming and outgoing particles, respectively. Anti-fermions are given by $v$ and $\overline{v}$ for incoming and outgoing particles, respectively. Incoming and outgoing photons are given by the polarisation vectors $\epsilon_\mu$ and $\epsilon_\mu^*$, respectively.

## A.2 Electroweak Theory

The electroweak interaction introduces many new rules to play with. The Feynman rules for electroweak theory are taken from [18] and [19]. The external lines, those representing ingoing and outgoing particles, are the same as QED. The propagators are given in Figure 7. The vertex couplings can be seen in Figure 8, where

$$g_W = \frac{e}{\sin(\theta_W)}, \quad V_{ji} = \text{CKM Matrix}, \tag{91}$$

relates to the $W$ boson couplings, and

$$g_z = \frac{e}{\sin(\theta_W)\cos(\theta_W)}, \quad v_f = I_{3f} - 2Q_f \sin^2(\theta_W), \quad a_f = I_{3f}, \tag{92}$$

relates to the $Z$ boson couplings.

There are additional couplings related to non-linear self-couplings, and Higgs interactions, but these are beyond the scope necessary for this report.

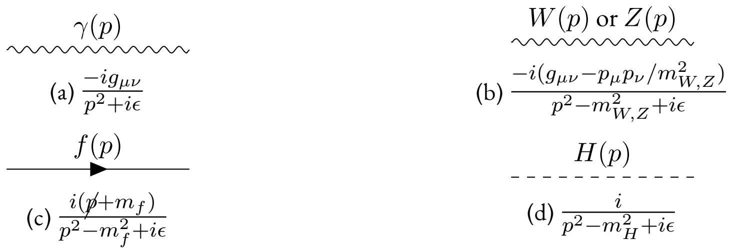


Figure 7: The propagators in electroweak theory.

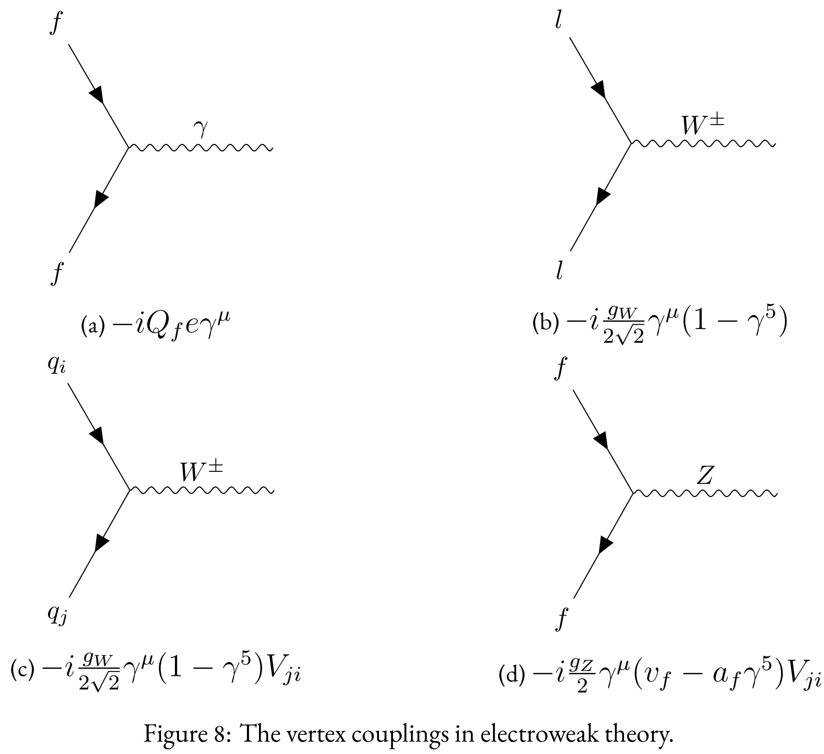


Figure 8: The vertex couplings in electroweak theory.

# B Dirac Algebra and Relations

This appendix contains some useful relations and simplifications for Dirac algebra, especially when dealing with numerators. Only expressions relevant to this report have been summarised below.

$$\sum_{r=1}^{2} u_r(\boldsymbol{p})\bar{u_r}(\boldsymbol{p}) = (\not{p} + m) \tag{93}$$

$$\sum_{r=1}^{2} v_r(\boldsymbol{p})\bar{v_r}(\boldsymbol{p}) = (\not{p} - m). \tag{94}$$

$$\sum \epsilon_\mu \epsilon_\nu^* \to -g_{\mu\nu}, \tag{95}$$

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0, \quad \overline{\psi} = \psi^\dagger\gamma^0. \tag{96}$$

$$(\overline{\psi}_a\gamma^\mu\psi_b)^* = \overline{\psi}_b\gamma^\mu\psi_a, \tag{97}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \tag{98}$$

$$i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix} = \gamma^5 \tag{99}$$

$$(\gamma^5)^2 = \mathbb{I}_2, \gamma^5, \gamma^\mu = 0, (\gamma^5)^\dagger = \gamma^5, \tag{100}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{101}$$

$$\gamma^\mu\gamma_\mu = 4\mathbb{I}_4, \tag{102}$$

$$\gamma^\mu\gamma_\mu\gamma^\nu = -2\gamma^\nu, \tag{103}$$

$$\gamma^\mu\gamma_\mu\gamma^\nu\gamma^\rho = 4g^{\mu\nu}, \tag{104}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho = g^{\mu\nu}\gamma^\rho + g^{\nu\rho}\gamma^\mu - g^{\mu\rho}\gamma^\nu - i\epsilon^{\sigma\mu\nu\rho}\gamma_\sigma\gamma^5, \tag{105}$$

$\epsilon^{\alpha\ldots\beta}$ is the Levi-Civita symbol in $n$ dimensions.

# C Additional Algebra

The following appendix contains additional detail to the derivations laid out in Sections 3 and 4.

## C.1 QED

In simplifying the numerator of some QED processes, namely the case of soft Bremsstrahlung in Section 3.1, there are a series of $\gamma$-matrices and 4-vectors in the numerator which require attention to simplify. The following two equations detail such simplifications

$$\begin{aligned}(\not{p}+m)\gamma^{\mu}\epsilon^*_{\mu}(k)u(p) &= (\gamma^{\mu}p_{\mu}\gamma^{\nu}\epsilon^*_{\nu}+m\gamma^{\nu}\epsilon^*_{\nu})u(p)\\ &= (\gamma^{\mu}p_{\mu}\gamma^{\mu}\epsilon^*_{\nu}+\gamma^{\nu}p_{\nu}\gamma^{\mu}\epsilon^*_{\nu}-\gamma^{\nu}p_{\mu}\gamma^{\mu}\epsilon^*_{\nu}+m\gamma^{\nu}\epsilon^*_{\nu})u(p)\\ &= (p_{\mu}\epsilon^*_{\nu}\{\gamma^{\mu},\gamma^{\nu}\}-\gamma^{\nu}p_{\mu}\gamma^{\mu}\epsilon^*_{\nu}+m\gamma^{\nu}\epsilon^*_{\nu})u(p)\\ &= (2p^{\nu}\epsilon^*_{\nu}+(-\not{p}+m)\gamma^{\nu}\epsilon^*_{\nu})\\ &= 2p^{\mu}\epsilon^*_{\mu}u(p),\end{aligned} \tag{106}$$

and so

$$\overline{u}(p')\gamma^{\mu}\epsilon^*_{\mu}(\not{p}'+m)=\overline{u}(p')2p'^{\mu}\epsilon^*_{\mu}. \tag{107}$$

## C.2 Electroweak

### C.2.1 Numerator Algebra

Likewise in QED, in the simplification of the numerator of some electroweak processes, namely soft Bremsstrahlung in Section 4.1, there are a series of $\gamma$-matrices and 4-vectors which require detailed analysis to simplify. This subsection is dedicated to providing detailed derivations of some algebraic simplifications.

With reference to (64), we look at a possible process for simplifying the numerator. Taking the first term, we can expand out the factor containing $v_f$,$a_f$, and $\gamma^5$, then collecting scalars and distributing the $\epsilon^*_{\mu}$ term,

$$\begin{aligned}(\not{p}+m)\Big(\frac{g_z}{2}\gamma^{\mu}(v_f-a_f\gamma^5)\Big)\epsilon^*_{\mu}u(p) &= \frac{g_z}{2}\Big((\not{p}+m)\gamma^{\mu}v_f-(\not{p}+m)\gamma^{\mu}a_f\gamma^5\Big)\epsilon^*_{\mu}u(p)\\ &= \frac{g_z}{2}\Big(v_f(\not{p}+m)\gamma^{\mu}\epsilon^*_{\mu}-a_f(\not{p}+m)\gamma^{\mu}\gamma^5\Big)u(p),\end{aligned} \tag{108}$$

where it is possible to see that the first term is the same as the QED case, with a scalar factor of $v_f$, and the second term will require more work. However, substitution of the QED case, from (106), enables additional clarity in seeing which steps to take next,

$$\frac{g_z}{2}\Big(v_f(\not{p}+m)\gamma^{\mu}\epsilon^*_{\mu}-a_f(\not{p}+m)\gamma^{\mu}\gamma^5\Big)u(p)=\frac{g_z}{2}\Big(2v_fp\cdot\epsilon^*-a_f(\not{p}+m)\gamma^{\mu}\gamma^5\Big)u(p). \tag{109}$$

$$(\not{p}+m)\gamma^{\mu}\gamma^5\epsilon^*_{\mu}u(p)=\gamma^5u(\not{p}-m)\gamma^{\mu}\epsilon^*_{\mu}u(p). \tag{110}$$

Now distributing the $\gamma^{\mu}$, using the anti-commutator identity, and applying the Dirac equation itself, we can simplify this expression down to its final form

$$\begin{aligned}\gamma^5u(\not{p}-m)\gamma^{\mu}\epsilon^*_{\mu}u(p) &= \gamma^5(\not{p}\gamma^{\mu}-m\gamma^{\mu})\epsilon^*_{\mu}u(p)\\ &= \gamma^5(\gamma^{\nu}p_{\nu}\gamma^{\mu}-m\gamma^{\mu})\epsilon^*_{\mu}u(p)\\ &= \gamma^5(p_{\nu}(2g^{\mu\nu}-\gamma^{\mu}\gamma^{\nu})-m\gamma^{\mu})\epsilon^*_{\mu}u(p)\\ &= \gamma^5(2p^{\mu}-\gamma^{\mu}\not{p}-m\gamma^{\mu})\epsilon^*_{\mu}u(p)\\ &= \gamma^5(2p^{\mu}-\gamma^{\mu}(\not{p}+m))\epsilon^*_{\mu}u(p)\\ &= \gamma^5(2p^{\mu}-\gamma^{\mu}(\not{p}-m+2m))\epsilon^*_{\mu}u(p)\\ &= \gamma^5(2p^{\mu}-\gamma^{\mu}(\not{p}-m)+2m\gamma^{\mu})\epsilon^*_{\mu}u(p)\\ &= \gamma^5(2p^{\mu}+2m\gamma^{\mu})\epsilon^*_{\mu}u(p)\\ &= 2\gamma^5(p^{\mu}+m\gamma^{\mu})\epsilon^*_{\mu}u(p).\end{aligned} \tag{111}$$

The other numerator can be simplified in a similar manner. The second numerator becomes

$$\overline{u}(p')(v_f + a_f\gamma^5)g_z p' \cdot \epsilon^*. \tag{112}$$

### C.2.2 Scattering Amplitude Simplification

Next we look at the simplification of the scattering amplitude, with reference to (64) after substitution of the simplifications made in Appendix C.2.1. The full expression for the amplitude under the new numerators is, and becomes,

$$\begin{aligned}
i\mathcal{M} &= \overline{u}(p')\left[\mathcal{M}_0 g_z(v_f - a_f\gamma^5)\frac{-p\cdot\epsilon^*}{2p\cdot k} + g_z(v_f + a_f\gamma^5)\frac{p'\cdot\epsilon^*}{2p'\cdot k}\mathcal{M}_0\right]u(p) \\
&= \frac{g_z}{2}\overline{u}(p')\left[\mathcal{M}_0(v_f - a_f\gamma^5)\frac{-p\cdot\epsilon^*}{p\cdot k} + (v_f + a_f\gamma^5)\frac{p'\cdot\epsilon^*}{p'\cdot k}\mathcal{M}_0\right]u(p) \\
&= \frac{g_z}{2}\overline{u}(p')\left[\mathcal{M}_0 v_f\frac{-p\cdot\epsilon^*}{p\cdot k} + \mathcal{M}_0 a_f\gamma^5\frac{p\cdot\epsilon^*}{p\cdot k} + v_f\mathcal{M}_0\frac{p'\cdot\epsilon^*}{p'\cdot k} + a_f\gamma^5\mathcal{M}_0\frac{p'\cdot\epsilon^*}{p'\cdot k}\right]u(p) \\
&= \frac{g_z}{2}\overline{u}(p')\left[\mathcal{M}_0 v_f\Big(\frac{p'\cdot\epsilon^*}{p'\cdot k} - \frac{p\cdot\epsilon^*}{p\cdot k}\Big) - \mathcal{M}_0 a_f\gamma^5\Big(\frac{p'\cdot\epsilon^*}{p'\cdot k} - \frac{p\cdot\epsilon^*}{p\cdot k}\Big)\right]u(p) \\
&= \frac{g_z}{2}\left(\overline{u}(p')\mathcal{M}_0(v_f - a_f\gamma^5)u(p)\right)\left[\frac{p'\cdot\epsilon^*}{p'\cdot k} - \frac{p\cdot\epsilon^*}{p\cdot k}\right].
\end{aligned} \tag{113}$$

### C.2.3 Hermitian Conjugate

The details of the simplification of (69) is derived below,

$$\begin{aligned}
\left(\overline{u}(p')\mathcal{M}_0 a_f\gamma^5 u(p)\right)^\dagger &= (u(p))^\dagger(\gamma^5)^\dagger a_f\mathcal{M}_0^\dagger(\overline{u}(p'))^\dagger \\
&= (u(p))^\dagger\gamma^5 a_f\mathcal{M}_0^\dagger((u(p'))^\dagger\gamma^0)^\dagger \\
&= -(u(p))^\dagger\gamma^0\gamma^5\gamma^0 a_f\mathcal{M}_0^\dagger((u(p'))^\dagger\gamma^0)^\dagger \\
&= -(u(p))^\dagger\gamma^0\gamma^5\gamma^0 a_f\mathcal{M}_0^\dagger\gamma^0 u(p') \\
&= -\overline{u}(p)\gamma^5 a_f\overline{\mathcal{M}}_0 u(p')
\end{aligned} \tag{114}$$

i.e. we have just picked up a minus sign.

### C.2.4 Trace Algebra

The trace algebra for (71) is detailed below

$$\begin{aligned}
\mathrm{Tr}(\overline{u}(p')\mathcal{M}_0 v_f u(p)\overline{u}(p)\mathcal{M}_0 v_f u(p')) &= \mathrm{Tr}(u(p')\overline{u}(p')\mathcal{M}_0 v_f u(p)\overline{u}(p)\mathcal{M}_0 v_f) \\
&= v_f^2\,\mathrm{Tr}(\not{p}'\mathcal{M}_0\not{p}\mathcal{M}_0) \\
&= v_f^2|\overline{u}(p')\mathcal{M}_0 u(p)|^2.
\end{aligned} \tag{115}$$

We can do the same for the second term, (72), taking care with the Dirac algebra and the $\gamma^5$'s,

$$\begin{aligned}
\mathrm{Tr}(\overline{u}(p')\mathcal{M}_0 a_f\gamma^5 u(p)\overline{u}(p)\mathcal{M}_0 a_f\gamma^5 u(p')) &= -\mathrm{Tr}(\not{p}' a_f\mathcal{M}_0\gamma^5\not{p}\gamma^5 a_f\mathcal{M}_0) \\
&= a_f^2\,\mathrm{Tr}(\not{p}'\mathcal{M}_0(\gamma^5)^2\not{p}\mathcal{M}_0) \\
&= a_f^2\,\mathrm{Tr}(\not{p}'\mathcal{M}_0\not{p}\mathcal{M}_0) \\
&= a_f^2|\not{u}(p')\mathcal{M}_0 u(p)|^2.
\end{aligned} \tag{116}$$

### C.2.5 Square Amplitude

The square of the amplitude, e.g. (70) can be derived by the following,

$$
\begin{aligned}
|\mathcal{M}|^2 = \frac{g_z^2}{4}\left[\frac{p'\cdot\epsilon^*}{p'\cdot k} - \frac{p\cdot\epsilon^*}{p\cdot k}\right]^2 \Bigg\{ &\overline{u}(p')\mathcal{M}_0 v_f u(p)\overline{u}(p)\overline{\mathcal{M}}_0 v_f u(p') \\
&+ \overline{u}(p')\mathcal{M}_0 a_f \gamma^5 u(p)\overline{u}(p)\overline{\mathcal{M}}_0 a_f \gamma^5 u(p') \\
&+ \overline{u}(p')\mathcal{M}_0 v_f u(p)\overline{u}(p)\overline{\mathcal{M}}_0 a_f \gamma^5 u(p') \\
&+ \overline{u}(p')\mathcal{M}_0 a_f \gamma^5 u(p)\overline{u}(p)\overline{\mathcal{M}}_0 v_f u(p') \Bigg\},
\end{aligned}
\tag{117}
$$

where the cross terms vanish.

Substituting the Hermitian conjugates from Appendix C.2.3 gives

$$
\begin{aligned}
|\mathcal{M}|^2 &= \frac{g_z^2}{4}\left[\frac{p'\cdot\epsilon^*}{p'\cdot k} - \frac{p\cdot\epsilon^*}{p\cdot k}\right]^2 \Bigg\{ \overline{u}(p')\mathcal{M}_0 v_f u(p)\overline{u}(p)\overline{\mathcal{M}}_0 v_f u(p') \\
&+ \overline{u}(p')\mathcal{M}_0 a_f \gamma^5 u(p)\overline{u}(p)\overline{\mathcal{M}}_0 a_f \gamma^5 u(p') \} \\
&= \frac{g_z^2}{4}\left[\frac{p'\cdot\epsilon^*}{p'\cdot k} - \frac{p\cdot\epsilon^*}{p\cdot k}\right]^2 \Bigg\{ v_f^2 |\overline{u}(p')\mathcal{M}_0 u(p)|^2 + a_f^2 |\cancel{u}(p')\mathcal{M}_0 u(p)|^2 \Bigg\} \\
&= \frac{g_z^2}{4}\left[\frac{p'\cdot\epsilon^*}{p'\cdot k} - \frac{p\cdot\epsilon^*}{p\cdot k}\right]^2 |\overline{u}(p')\mathcal{M}_0 u(p)|^2 (v_f^2 + a_f^2).
\end{aligned}
\tag{118}
$$