Models of Compact Stars

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Abstract

By using an appropriate approximation for the equation of state for the interior fluid in compact stars, their structural equations can be solved using numerical methods to find their radii and masses. These structural equations arise from the necessity for hydrostatic equilibrium to hold in order for these stars to exist stably. The boundary conditions required to solve these differential equations are the central pressure in the star and the mass in the centre (~ 0 kg). Solutions for the radius and mass as a function of central pressure were found for both white dwarfs (briefly) and neutron stars (in detail). We show the radii and mass functions for neutron stars, including general relativistic corrections and treating the interior structure as an ideal gas of neutrons. This model assumes neutron stars are comprised of neutrons only, and so an extension is also included which includes electrons and protons in this model. Both models predict that the most massive stable neutron star has a mass of 1.41×10^{30} kg ($0.702 M_{\odot}$).

1 Introduction

Compact stars is the term used to describe stars which are in the final stages of evolution in their lifetimes. Examples of these are neutron stars and white dwarfs. These stars have very high pressures and densities, meaning that their properties are extreme. The internal structure of these stars is made from a plasma of subatomic particles; the behaviour of this plasma is complex and very difficult to predict. Therefore, certain approximations for the equations of state of this fluid must be used. Additionally, due to the extremely high density of these stars, the gravitational behaviour is also very complex. Newtonian gravity can be used to estimate the gravitational behaviour, but relativistic corrections must be considered for a much more accurate representation - especially at higher pressures. By applying approximations of the behaviour of the fluid within these stars along with the specified equations of gravity to the equation of hydrostatic equilibrium, certain properties of these compact stars can be estimated (such as mass and radius). It is very difficult to solve these differential equations with analytic methods, and so numerical methods are employed. We explore how these properties vary with the central pressure of different stars. We also note that even the relativistic solutions for mass and radius still include certain approximations, namely that these stellar objects are both non-rotating and non-magnetized [2]. We note that the structure for this paper closely follows Ref. [8]. However, the details of our method vary in places and we have also included an extra section on discontinuities not contained in the original paper.

2 Models of Gravity for Compact Stars

In order for compact stars to exist stably, the net internal forces in the star must balance exactly. The most prominent forces are pressure due to nuclear fusion or degeneracy pressure, and the self-gravity. In compact stars, the internal pressure is due to the Pauli exclusion principle, which prohibits fermions from occupying the same quantum state due to the asymmetry of the Schrodinger wavefunction [3]. This force is then balanced by the force of gravity, which arises due to the interior mass of the star. Because the density of compact stars is so vast, these forces have huge magnitudes, and it is this extreme density that leads to their complex properties. However, by using hydrostatic equilibrium equations defining these properties can be found. The first of the equations can be found by simply modelling the star as a sphere and considering the mass in a spherical shell. This gives rise to the equation [8]:

$$\frac{dm(r)}{dr} = \frac{4\pi r^2 \epsilon(r)}{c^2}, \tag{1}$$

where ϵ is the energy density and is equal to $\rho(r)c^2$ - the mass density multiplied by c^2 and $m(r)$ is the mass at radius r. To find the equation for pressure an appropriate gravitational model must be included.

We note that due to the complexities of solving the structural differential equations mentioned in this section using analytically methods, numerical methods are used instead. This makes it much simpler to compute the solution and, therefore, saves a significant amount of time. The main method we use for the computations is one of the Runge-Kutta methods (RK) - in particular the RK-45 method, where the '4' originates from the initial perturbation order. RK methods utilize the rate of change from the differential equation to approximate the increment in each step (this being radius in our case), and then to obtain the final state at the end of each step. These methods originate from the Euler method [12].

2.1 Newtonian Gravity

Newton's law of gravitation was originally derived from empirical observations. It stated that there is a force between any two particles in the universe that is proportional to the product of their masses over the distance between them squared. Newton's law of gravity is widely known, but we take it from Ref. [1]

$$F = \frac{GMm}{R^2}. \tag{2}$$

Here,F is the force, G is the gravitational constant, M and m are the particle masses and R is the distance between the particles. More recent research has found Newtonian gravity can be explained as a product of the temperature and change in entropy being equal to the work done by the gravitational force [10]. By combining

Eqn. (2) with Eqn. (1) and equating it to the pressure equation , as hydrostatic equilibrium applies, we obtain [8]:

$$\frac{dp(r)}{dr} = -\frac{-G\epsilon(r)m(r)}{c^2r^2}, \tag{3}$$

where p is the pressure, and all other symbols have the same meaning as described earlier. Eqn. (3), along with Eqn. (1), form a set of coupled differential equations that can be solved, given initial boundary conditions - namely the central pressure and central mass. However, there is a complication involved. For the pressure at each point in the star, the corresponding energy density must be found. This problem can be solved by using an appropriate equation of state to approximate the fluid in the stars. We explore this in detail in Section 3. While Newtonian gravity does provide a simplistic view of the gravitational effects in and around compact stars, it is still useful to give a fairly accurate depiction of the gravitational properties of compact stars of lower central pressures and masses. Therefore, this model was used as a good starting block before building in the relativistic corrections.

2.2 Relativistic Gravity

Relativistic gravity is in essence just a generalisation of Newtonian gravity that includes the effects of curved spacetime. For very compact stars, when the gravitational forces become far more extreme, the corrections applied to the Newtonian model provide a far more accurate depiction of the system. We will not go into detail about general relativity as that is beyond the scope of this paper, instead, we will simply provide the relevant equation [8]

$$G_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu}, \tag{4}$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the stress-energy tensor. This equation can then be applied to the equations for hydrostatic equilibrium to give the Tolman-Oppenheimer-Volkoff (TOV) equation [8]:

$$\frac{dp}{dr} = -\frac{G\epsilon(r)m(r)}{c^2r^2}\left[1+\frac{p(r)}{\epsilon(r)}\right]\left[1+\frac{4\pi r^3 p(r)}{m(r)c^2}\right]\left[1-\frac{2Gm(r)}{c^2r}\right]^{-1}. \tag{5}$$

This equation applies to an isotropic, general relativistic, static ideal fluid sphere in hydrostatic equilibrium. These corrections become very important for stars with very high central pressure, as the gravitational forces are much more significant and extreme, and so a Newtonian model cannot accurately predict the behaviour.

3 Equations of State

3.1 Ideal Gas of Fermions

Since neutrons and electrons are fermions, we can consider the equation of state for an ideal gas of fermions to describe a fluid of degenerate neutrons or electrons within a neutron star or white dwarf. Like other equations of state, it provides a description of the relationship between pressure and energy density or other analogous variables, such as mass density. We will give the equation of state here without derivation. The equation of state is given by [8]:

$$\epsilon_i(x_i) = \frac{\epsilon_0}{8}[(2x_i^3 + x_i)(1+x_i^2)^{\frac{1}{2}} - \sinh^{-1}(x_i)], \tag{6}$$

and

$$p_i(x_i) = \frac{\epsilon_0}{24}[(2x_i^3 - 3x_i)(1+x_i^2)^{\frac{1}{2}} + 3\sinh^{-1}(x_i)], \tag{7}$$

where

$$x_i = \frac{k_i}{m_i c},\ \epsilon_0 = \frac{m_i^4 c^5}{\pi^2 \hbar^3},\ i = n, p, e. \tag{8}$$

Where x_i is a dimensionless variable proportional to k_i, the fermi momentum, i is the placeholder for a fermion (e - electron, n - neutron, ...), and m_i is the particle mass for each fermion i. All other symbols have the same meaning as given previously. This equation of state assumes that the particles do not interact, since it still behaves like an ideal gas, but factors additional pressure from to degeneracy pressure. For pure neutron stars we use $i = n$

only. Although calculating x_n from first principles would allow us to describe the pressure and energy density of the gas, there is no mechanism for us to calculate this within the system we are describing. Instead, we can use a root-finding algorithm on $p(x_n) - p = 0$ to find the approximate value of x_n at the current radial coordinate r. Substituting this value of x_n into $\epsilon(x_n)$ provides the energy density at the radial coordinate, hence decoupling the equations of hydrostatic equilibrium. For the case where we consider an ideal gas of pure neutrons, we use the bisect method of root-finding for its rapid convergence to solve the eqution $p(x_n) - p = 0$.

In the non-relativistic, and ultra-relativistic limits of Eqn. (6) and Eqn. (7) for one species, the equation of state becomes polytropic and no minimisations are necessary since it is analytic. We discuss polytropic equations of state in more detail in Section 3.3.

3.2 Simplified Ideal Gas of Protons, Neutrons, and Electrons

Since neutrons are unstable, they undergo random β decay, forming a proton, electron, and anti-electron neutrino. In addition, protons and electrons may collide and form a neutron and electron neutrino. These reactions exist within an equilibrium of the two equations described below:

$$p + e^- \to n + \nu_{e^-}, \tag{9}$$

$$n \to p + e^- + \overline{\nu}_{e^-}. \tag{10}$$

This suggests that neutron stars will be composed of protons and electrons at some pressures. Again, we will give the equation of state without derivation, which in this case is simply the sum of the individual equations of state, each having its own value of x_i [8]:

$$\epsilon_{tot} = \sum_{i=n,p,e} \epsilon_i, \;\; p_{tot} = \sum_{i=n,p,e} p_i. \tag{11}$$

As first mentioned in the previous section, x_i can be found by minimisation/root-finding, but since $i = n, p, e$ we will have to root-find in 3 dimensions. Since each i is a fermion, ϵ_i and p_i are analogous to Eqn. (6) and Eqn. (7) respectively, only differing in the mass variable m_i. We can modify the value of ϵ_0 for each i, and the form of the rest of each ϵ_i and p_i remains the same otherwise. The method of minimisation needs some small modifications to allow two additional roots to be found. We now have $p(x_n, x_p, x_e) - p = 0$, and hence the bisect-method is no longer viable. For this project, we use Broyden's method, which allows for root-finding using multiple variables. As a result of using a 3-variable minimisation, the computation time for finding x_i is massively inflated. To simplify this somewhat, we introduce some further approximations. The full summation (Eqn. (11)) predicts a pressure p_{crit} - the pressure below which no neutrons exist, as the equilibrium is favoured immensely towards the neutrons decaying, rather than protons and electrons forming new neutrons. We find $p_{crit} = 3.038 \times 10^{23}$ from Ref. [8]. As such, 3-variable minimisation is only necessary above p_{crit}. Below p_{crit}, we use 2-variable minimisation - introducing a small discontinuity into the equation. This is akin to saying that the system obeys only Eqn. (10). This is still immensely computationally intense, however. We further simplify the equation of state by postulating that above p_{crit} only neutrons exist (not a mixture of all three), and that the system obeys only Eqn. (9). We assume that the effect of the neutrinos is negligible. This reduces the problem to a 1-variable minimisation above p_{crit} and 2-variable minimisation below p_{crit}. Thus, the equation of state becomes

$$\epsilon_{tot} = \begin{cases} \epsilon_n & \text{for } p > p_{crit} \\ \sum_{i=p,e} \epsilon_i & \text{for } p < p_{crit} \end{cases}, \; p_{tot} = \begin{cases} p_n & \text{for } p > p_{crit} \\ \sum_{i=p,e} p_i & \text{for } p < p_{crit} \end{cases}. \tag{12}$$

Nonetheless, this still takes on average 5 mins (by multithreading with 6 cores) to compute the radial pressure and mass functions for one star. We discuss some computational techniques later in Section 6.1.

3.3 Polytropic Equation of State

As mentioned earlier we can take the non-relativistic and relativistic limits of some equations of state to give simple polytropic equations of state of the form

$$p(\epsilon) = K\epsilon^{\gamma}, \tag{13}$$

using $\epsilon = \rho c^2$. Where K is the polytropic constant, with the same units as pressure, and γ is the polytropic index and relates to which relativistic limit is used. K also differs for different limits. By changing the value of K and γ the relativistic and non-relativistic cases can be analysed. When this is subbed into Eqn. (1) and Eqn. (3), the structural equations are simplified to the following [8]:

$$\frac{dp(r)}{dr} = -\frac{R_0 p(r)^{1/\gamma}\overline{m}(r)}{r^2 K^{1/\gamma}} \tag{14}$$

and

$$\frac{d\overline{m}(r)}{dr} = \frac{4\pi r^2}{M_\odot c^2}\left(\frac{p(r)}{K}\right)^{1/\gamma}. \tag{15}$$

In this case, R_0 is the radius of the sun, and the overbar on m is used to symbolise that it m is now dimensionless, since it has been scaled by $\frac{1}{M_\odot}$, the mass of the sun. From Ref. [8], we find for an ideal gas of fermions (i), the limits of the equations have constants

$$K_{i,non-rel} = \frac{\hbar^2}{15\pi^2 m_i}\left(\frac{3\pi^2 Z}{m_i c^2 A}\right)^{\frac{5}{3}}, \tag{16}$$

and

$$K_{i,rel} = \frac{\hbar}{12\pi^2}\left(\frac{3\pi^2 Z}{m_i c^2 A}\right)^{\frac{4}{3}}, \tag{17}$$

where Z/A is the atomic number to atomic mass ratio, and is 0.5 for white dwarfs predominately made from electrons and Carbon-12 nuclei, and 1 for neutron stars made from pure neutrons. This is useful for modelling white dwarfs as ideal gasses of degenerate electrons. We can solve the structural equations much more easily without having to minimise functions of one or more variable(s). We discuss solving the full structural equations in the following sections.

4 White Dwarfs

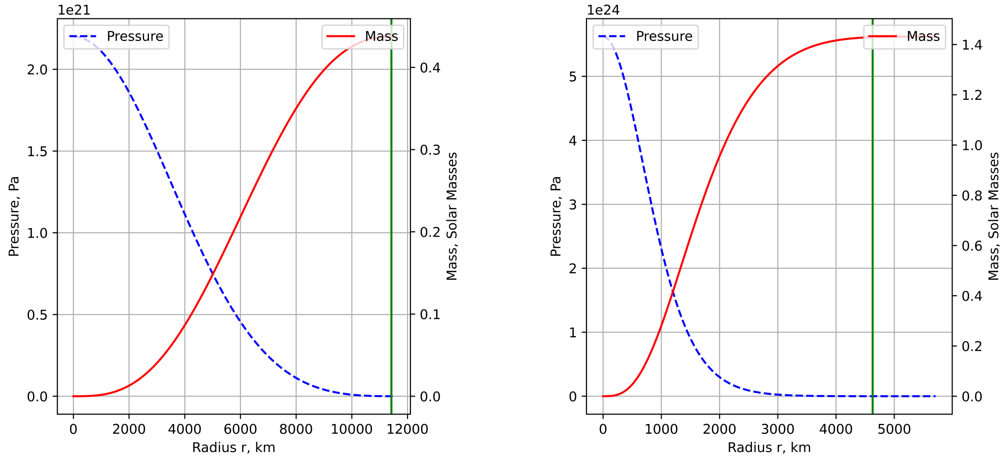
Before discussing neutron stars in full, we first look at white dwarf stars, which are also compact stars but are not as complex as neutron stars. The main difference in the complexity is due to the difference in pressure; neutron stars have significantly higher central pressures than white dwarfs. This means that Newtonian models do not accurately model neutron stars at higher pressures, whereas they can be used to model white dwarfs fairly accurately.

4.1 Solving the Structural Equations

Before varying the pressure to investigate the properties over a range of pressures, one must first solve Eqn. (1) and Eqn. (3) to find a white dwarf's mass and radius for a given initial pressure. We distinguish between the relativistic and non-relativistic cases by setting $\gamma = 4/3$ and $\gamma = 5/3$ respectively. To clarify, the inclusion of relativity concerns the motion of the particles, not the gravitational behaviour, which is still purely Newtonian. As mentioned earlier the problems with solving Eqn. (1) and Eqn. (3), are solved by using the polytropic equation of state. So we are actually solving Eqn. (14) and Eqn. (15), which is comparatively very straightforward. By using the SciPy Solve IVP method (RK-45), they can be solved easily for a given initial central pressure, central mass and range of radii [11]. For white dwarfs, we set the central mass to zero, and arbitrarily choose the central pressure to investigate different stars.

Figure 1 (a) shows the curve for the non-relativistic case where the initial pressure $p_0 = 2.33 \times 10^{22}$ dyne/cm^2. Figure 1 (b) shows a similar graph for the relativistic case, where the initial pressure $p_0 = 5.26 \times 10^{24}$ dyne/cm^2. The initial pressures must be changed along with the value of γ in order to differentiate between the relativistic and non-relativistic cases. These plots agree very nicely with the equivalent plots (Figures 1 and 2) in Ref. [8].

In order to find the minimum pressure, a root-finding algorithm must be used to find the value accurately. Once this value is found, the mass and radius of the star can be easily determined by simply reading their values from the corresponding pressure and mass columns in the data. We programmed a root-finding algorithm in Python to find the pressure at which we consider the edge of the star to be. The algorithm calculated the gradient between



(a) Non-Relativistic

(b) Relativistic

Figure 1: The mass and pressure as a function of radius for a non-relativistic white dwarf with initial pressure $p_0 = 2.20 \times 10^{22}$ dyne/cm^2 (a) and for a relativistic white dwarf with initial pressure $p_0 = 5.62 \times 10^{24}$ dyne/cm^2 (b).

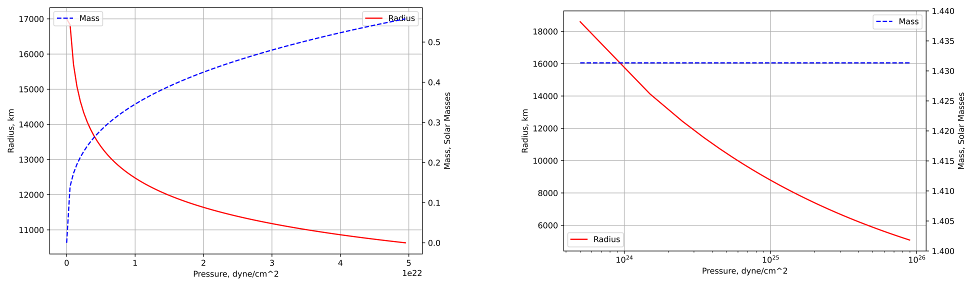
successive points until the gradient was less than or equal to some tolerance. We define the edge of the star at this boundary, and the pressure value is determined. The user can define an input tolerance that is multiplied by the gradient between the first and last points on the pressure curve to give the tolerance that the algorithm compares against. This is done for both the relativistic and non-relativistic cases. For the non-relativistic case, the radius is found to be 1.14×10^4 km and the mass is found to be 7.96×10^{29} kg or $0.398 M_\odot$. For the relativistic case, the radius is found to be 4.63×10^3 km and the mass is found to be 2.78×10^{30} kg or $1.39 M_\odot$. These values are in agreement with those in Ref. [8], with only a small discrepancy likely due to the difference in our method. The next step is then to look at these values for many different stars by varying the central pressure.

4.2 Varying Central Pressure

Now that we have a way to find the solutions for one central pressure, we vary the central pressure to look at mass and radii vary for many different white dwarfs. To do this we use the same method as described previously and use the gradient-based root-finding method to calculate the radius and mass for a given central pressure. We then add new code to vary the central pressure in increments, allowing us to plot the edge-of-star radii and masses against central pressure. The method for this process was not difficult to code on its own. However, the main issue was the time taken for the code to fully run. Therefore, we had to consider optimising our code as best as possible. This included using for loops rather than while loops, for example, as they are more efficient, which is due to the fact while loops must check if the given condition is satisfied after each iteration. Using these methods, we are able to produce Figures 2 (a) and (b). Once again these figures agree closely with the plots from Ref. [8] as well as theoretical predictions ,thus, validating our method. We also note that a logarithmic scale is used for Figure 2 (b). This method works well for white dwarfs, however for neutron stars, it is not sufficient. With the inclusion of TOV as well as eventually dropping the polytopic equation of state estimation, the equations are not as easily solved. These problems will be discussed in the following sections.

5 Pure Neutron Stars

Whilst both are compact stars, white dwarfs and neutron stars differ significantly in many aspects. The average white dwarf has a mass of $0.5\ M_\odot$ and is 6000 km in radius [4]. Whereas, neutron stars tend to be $1.4\ M_\odot$ in mass with a radius of 10 km [9]. For the purposes of this project, we consider the only mathematical difference



(a) Non-Relativistic (b) Relativistic

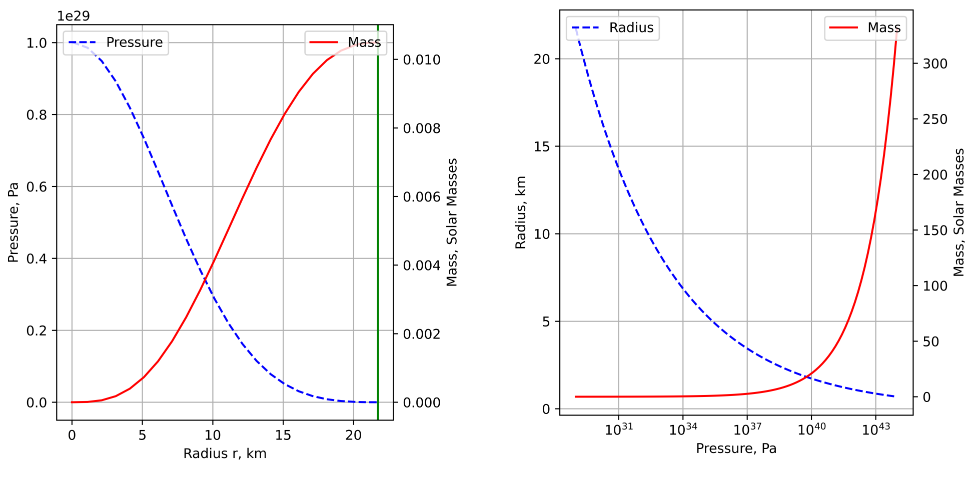
Figure 2: Calculated masses and radii for white dwarfs of central pressures up to 5×10^{22} dyne/cm^2 for the non-relativistic polytope (a) and central pressures between 5×10^{23} dyne/cm^2 and 9×10^{25} dyne/cm^2 for the relativistic polytope (b).

between the two types of stars to be the equation of state (i.e. their composition and how their composition interacts with itself). This section explores the different equations of state for neutron stars and introduces the Tolman-Oppenheimer-Volkoff (TOV) equation as a relativistic correction to the Newtonian equations of hydrostatic equilibrium.

5.1 Newtonain Models

5.1.1 Neutron Stars as a Polytropic Gas of Neutrons

In this subsection, we continue to use Newtonian gravity with a polytropic equation of state, assuming that the particles are non-relativistic in motion. Hence, we will use the value of $K_{non-rel}$ as given in Eqn. (16).



(a) Single Star (b) Range

Figure 3: (a) radial density and pressure functions of a neutron star with central pressure $p_0 = 1.0 \times 10^{29}$ Pa. The model is limited by Newtonian gravity and $K_{non-rel}$. The green line represents the edge of the star at radius 21.7 km with mass 1.99×10^{28} kg ($0.01 M_\odot$). (b) stellar radius and mass as functions of central pressure in Pa for 100 different stars of central pressures spaced logarithmically between 1×10^{28} and 1×10^{44} Pa.

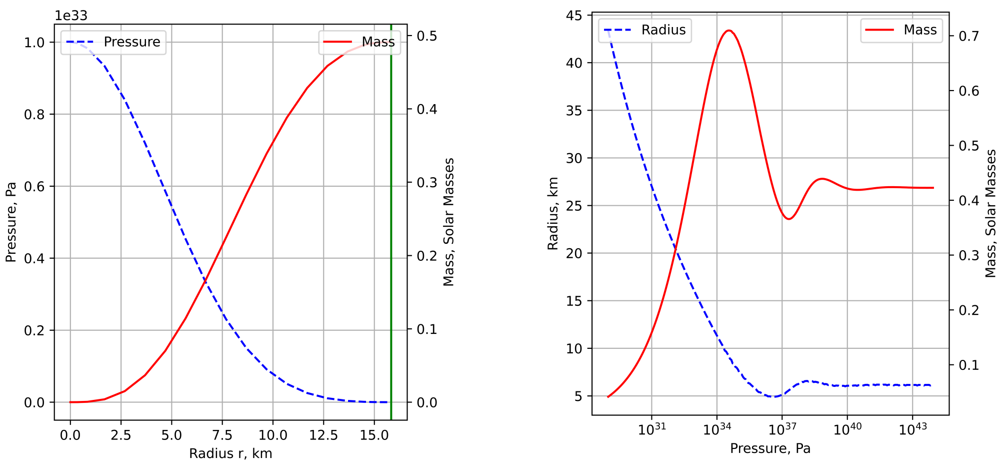
As mentioned in Section 3, each value of K forms a pair with a corresponding polytropic index. In this case

$\gamma_{non-rel} = 5/3$ is the corresponding γ for $K_{non-rel}$. Substituting these into Eqn. (15) and Eqn. (14), we arrive at a pair of differential equations ready to be solved numerically. Again, we choose to use the RK-45 method for its efficiency, and high accuracy in the number of dimensions we're modelling. Figure 3 (a) shows the radial pressure and mass functions for a neutron star of central pressure 1×10^{29} Pa modelled under these conditions. We see that the shape of Figure 3 (a) is similar to Figure 1 since they share the same differential equations, and only differ by a scalar change in K. The edge of the star is defined by the gradient-based root-finding method we described earlier. For the star in Figure 3, the edge is defined at a radius of 21.7 km with a mass of 1.99×10^{28} kg ($0.01 M_{\odot}$). Figure 3 (b) shows the stellar radius and mass as a function of central pressure. It is more than noteworthy that this function appears to have no maximum within the expected range of central pressures for a neutron star. This is clearly incorrect, as this would suggest that there is no limit to degeneracy pressure, and in turn, we may not have a mechanism to collapse neutron stars into black holes. Instead, it suggests that neutron stars would continue to accrete mass indefinitely, and this is not what we observe [7]. It is impossible to consider the relativistic case of particles in neutron stars, as the pressure does not converge to 0, and hence the radius of the star would be calculated as 'infinite', which is clearly incorrect [8]. Thus, we move on to more advanced models.

5.2 Fully Relativistic Models

Up until now, we have been assuming a Newtonian model of gravity. In this section, we introduce relativistic corrections to the equation of hydrostatic equilibrium as first described in Eqn. (5). The modifications of TOV only apply to the differential for pressure (naturally), and the mass differential remains the same as first described. It is still necessary to use an equation of state to describe the full system of the interior fluid of the neutron star, as otherwise, we would not have given as many equations as there are unknown variables. One consideration we must make when dealing with Eqn. (5), is that we may no longer use the initial condition that $m(0) = 0$, since at $r = 0$, we would be dividing by 0, which has no defined value. Instead, we change the boundary condition to something arbitrarily small, in our case we choose to use $m(0) = 1 \times 10^{-6}$ kg.

5.2.1 Neutron Stars as an Ideal Gas of Neutrons



(a) Single Star (b) Range

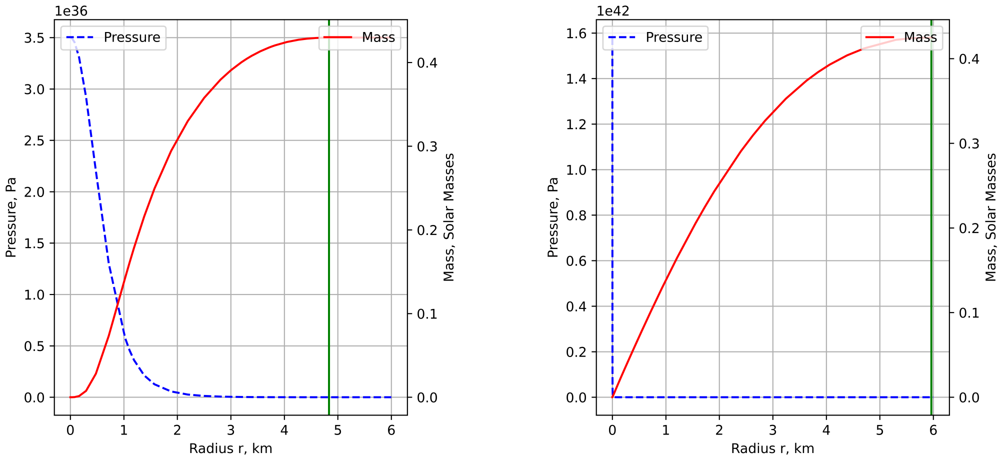
Figure 4: (a) pressure (dashed blue) and enclosed mass (red) functions for a single neutron star composed of an ideal gas of pure neutrons with central pressure 1×10^{33} Pa. The green line represents the edge of the star with a radius of 15.8 km and mass 9.76×10^{29} kg ($0.49 M_{\odot}$). (b) neutron star radius and mass functions for a range of 250 central pressures (points) between 1×10^{28} Pa and 1×10^{44} Pa. The maximum stable mass is 1.41×10^{30} kg ($0.71 M_{\odot}$) at with a radius of 9.22 km and central pressure 3.48×10^{34} Pa.

Briefly, we will consider the full ideal gas of neutrons (fermions) as the composition for the equation of state, i.e. using Eqn. (6) and Eqn. (7) from earlier. We apply the equation of state to Eqn. (5). This provides us with a complete set of differential equations to solve. We continue to use the RK-45 method of solving differential equations and opt to use the SciPy Solve IVP method specifically to achieve this [11]. We use the bisect-method of root finding to find the value of x that minimises $p(x) - p = 0$ and gives ϵ. Figure 4 shows the key radial mass and pressure functions for a neutron star of central pressure 1×10^{33} Pa composed of pure neutrons acting as an ideal gas of fermions. We continue to use a gradient-based root-finding function to determine the edge of the star, and subsequently provide the radius and mass of the star. Figure 4 (b) agrees with Figure 9 in Ref. [8]. Ref. [8] states that the most massive stable neutron star according to this model has a mass of 1.42×10^{30} kg $(0.71 M_{\odot})$, a radius of 9.14 km with central pressure 3.5×10^{34} Pa. The differences between our calculations and those in Ref. [8] are negligible, and so we support the findings of Sagert, I. et al in this regard.

6 Mixed Composition Neutron Stars

As an extension of the project, we consider other particles that may affect the pressure within a neutron star, namely protons and electrons.

6.1 Neutron Stars as an Ideal Gas of Neutrons, Protons, and Electrons

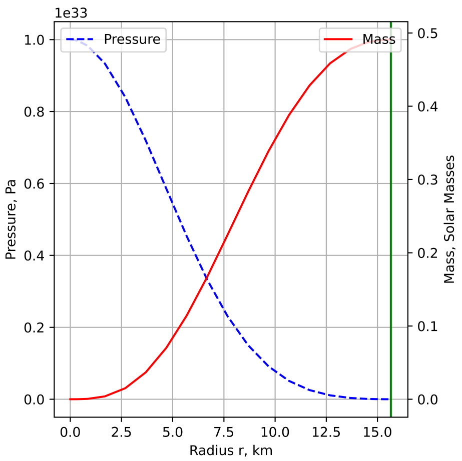


(a) Moderate Central Pressure, $p_0 = 3.50 \times 10^{36}$ Pa

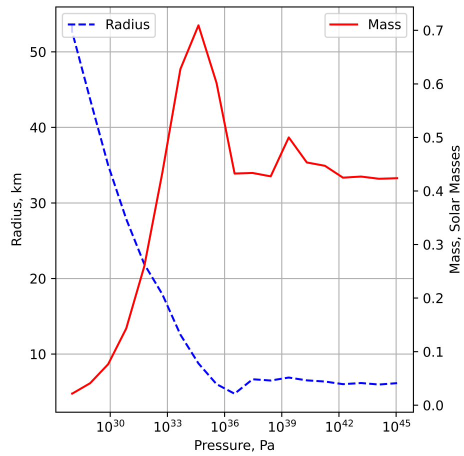
(b) High Central Pressure (Linear), $p_0 = 1.58 \times 10^{42}$ Pa

Figure 5: Enclosed mass (red) and pressure (blue, dashed) as functions of radius for neutron stars composed a simplified ideal gas of protons, neutrons, and electrons with a central pressure $p_0 = 3.50 \times 10^{36}$ Pa (a) and $p_0 = 1.58 \times 10^{42}$ Pa (b). The green line represents the edge of the star. (a) has a radius of 4.83×10^3 km and mass of 8.55×10^{29} kg $(0.43 M_{\odot})$. (b) has a radius of 5.96×10^3 km and mass of 8.35×10^{29} kg $(0.43 M_{\odot})$.

As laid out in Section 3.2, we only consider a simplified form of the effect from additional particles. Since multivariable optimisation is so computationally intense, we wrote some additional 'helper functions' to accelerate the process of finding $\epsilon(x_n, x_p, x_e)$ for each corresponding $p(x_n, x_p, x_e)$. We developed a function that stores each value of ϵ and p in a file when they are first calculated. Then for each subsequent 'calculation' where we may be referring to the same ϵ and p, the program would search the file for these values first. It is possible for the user to define a tolerance, whereby the search function would accept any value of ϵ that was in a range of $p \pm tol$, where tol is the tolerance. We define the tolerance to be 1×10^{15} Pa, as this is insignificant compared to the scales of the pressures we are solving for, but is large enough to provide faster computation times. This, however, limits the resolution of our calculations to tol at the absolute minimum. As such, we must consider any changes less than tol to be insignificant, and any pressures less than tol as null. We find this function is still faster than the SciPy Optimise function, even when the number of pairs to search through exceeds 200,000. In addition to this, we multithreaded the for loops to work using all logical cores, as opposed to just one. We briefly considered using CUDA programming to run the loops on a GPU (>10,000 logical cores), but this task was deemed too time-consuming for this project. In total, we pre-calculated 229,704 pairs of ϵ and p. The 'computation time' for solving the structural equations for a single star went from approximately 6 hrs to 0.5 s on average [6]. Apart from the additional helper functions mentioned and the new boundary condition described at the start of Section 5.2, we calculate the functions for the star in the same way as set out in previous sections. Figure 5 shows how the structure of the stars begins to change as central pressure increases. This effect is entirely different to any functions we have seen previously, where all functions have had the same shape but simply 'scaled' differently. Figure 5 (a) shows that the rate at which pressure decreases from the initial setting is much greater compared to earlier functions. This effect is more pronounced in Figure 5 (b) where the drop in pressure is extreme. This does happen to the mass function, but not to the same extent. It is visible that the second derivative of the mass function at the origin is greater at higher central pressures.



(a) Single Star



(b) Range

Figure 6: (a): Pressure (dashed blue) and enclosed mass (red) functions for a single neutron star composed of simplified ideal gas of protons, neutrons, and electrons with central pressure 1×10^{33} Pa. The green line represents the edge of the star with radius 15.7 km and mass 9.75×10^{29} kg ($0.49 M_{\odot}$). (b): Neutron star radius and mass functions for a range of 20 stars with central pressures between 1×10^{28} Pa and 1×10^{44} Pa. The maximum stable mass is 1.41×10^{30} kg ($0.71 M_{\odot}$) at with a radius of 8.76 km and central pressure 4.28×10^{34} Pa.

We believe that these graphs indicate layers and structures within the star. It is indicated by Figures 5 (a), and (b) that as central pressure increases, two distinct layers appear to manifest within the star. One is a core of dense, high-pressure material (neutrons since $p > p_{crit}$), which is very small in diameter. The other is a very large crust of low-pressure material (electrons and protons as $p < p_{crit}$). This, however, is to be expected as we have specified this in the equation of state by introducing a discontinuity. Therefore, we cannot conclude that the layers truly exist, since they were a postulate used to produce our equation of state. However, we may still comment on whether or not including mixed composition in the equation of state changes the predicted mass for the largest stable neutron star. To determine the edge of the star, we modify our gradient-based root-finding method to use the mass function rather than the pressure function. Using the pressure function would likely give a smaller radius and mass than expected since it appears to 'flatten' before the mass function has saturated. We note that there is no equivalent plot in Ref. [8], so we cannot compare our results to this paper - most likely since we have made different approximations regarding the equation of state for neutron stars of mixed composition (see Section 3.2). 6 (a) shows the mass and pressure functions for a single star of central pressure $p_0 = 1 \times 10^{33}$ Pa composed of a simplified ideal gas of protons, neutrons, and electrons. It has a mass of 9.75×10^{29} kg ($0.49 M_{\odot}$) and a radius 15.7 km. We compare this to Figure 4 (a) which has a mass of 9.76×10^{29} kg ($0.49 M_{\odot}$) and radius 15.8 km and is composed of an ideal gas of neutrons only. Surprisingly, there is little difference between the overall mass and radius for the two models, only differing by 0.01 units for each characteristic. Figure 6 (b) shows the masses and radii of neutron stars composed of protons, neutrons, and electrons. The maximum stable mass is 1.41×10^{30} kg $(0.71 M_{\odot})$ with a radius of 8.67 km. This differs only marginally from the neutron-only model, which predicts the same mass with a larger radius of 9.22 km. We were only able to compute 20 central pressures as the limits of integration for the SciPy Solve IVP had to be set manually, and determined individually for each central pressure to ensure that the integration did not 'run away'. On many occasions, the pressure did not converge to 0. In many cases, the pressure plateaued at $p \sim 1 \times 10^{23}$ Pa, which we deemed low enough to be considered acceptable grounds to terminate the integration, as this is minuscule compared to the initial pressure, and the conditions for our gradient-based root finding method were still met. Figure 6 (b) does not agree with Figure 12 in Ref. [8]. This is likely becuase we have made different approximations to Ref. [8] regarding the equation of state for neutron stars of mixed composition. Ref. [8] allows a mixture of protons, neutrons, and electrons to exist above p_{crit}. Whereas, for computational efficiency, we chose to assume that only neutrons exist above p_{crit}. Therefore, we are not plotting the same system as Ref. [8]. However, we note that Figure 4 (b) is visually similar to Figure 6 (b), and both have the same mass maxima. Since we assume that neutrons only exist for pressures above p_{crit} for our mixed composition model, the equation of state for both the mixed composition model and pure neutron model will be identical above p_{crit}.

6.2 Conclusion: Limit, Black Holes, and Collapse

In conclusion, as mentioned earlier, the maximum stable mass we predict for a neutron star is 1.41×10^{30} kg $(\sim 0.702 M_{\odot})$ with a radius of 8.76 km. We believe that above this mass, the neutron star would collapse into a black hole since the gravitational force would exceed the degeneracy pressure of the neutrons in the core, and the degeneracy pressure of the protons and electrons in the crust. However, the current largest known stable neutron star (PSR J0952–0607) has a mass of $2.35 \pm 0.17 M_{\odot}$, which greatly exceeds our maximum [7]. Thus, it is likely that Eqn. (12) is not correct in reality. We question whether using Eqn. (11) as the equation of state, without simplifying, would suggest a maximum stable mass greater than PSR J0952–0607. However, since we have not included the effect of magnetism or rotation in our theory of gravity, we cannot criticise our model endlessly for this inaccuracy. That being said, for future work on this topic, the next step would be to find and use appropriate models that include these effects, to more accurately model compact stars. Other papers that use similar approximations suggest that the maximum stable mass a star can be due to the force of neutron degeneracy pressure is about $0.7 M_{\odot}$, which is about the same as our prediction [5]. Therefore, although it is clear that we have not considered an appropriate equation of state or model of gravity, leading to the discrepancy from what is physically observed. Our calculations agree strongly with other research papers (primarily, Ref. [8]) suggesting our method is most likely correct for the model we have chosen. Our codebase is publicly available on GitHub from Ref. [6].

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